Numerical Analysis — Problem Set 11 — Differential Equations

Due Thursday, Dec. 1 (beginning of class)

This problem set does extremely sophisticated stuff based on the quite simple theory and program we worked through last week.

The idea is to: (a) give you two additional examples to illustrate the theory, and (b) to give you a sense of how a real field that you might be interested in (the field of mathematical ecology) can use the techniques to find solutions to ecological models.

It will be a good challenge to code these differential equations in lines 18-30 of the program on p. 83, which is all you have available. If you can't do it in only 13 lines, our HP-25 emulation actually allows for more than 49 steps, unlike a real HP-25.

1. The Allee Effect

In ecology, the population of a species is an interesting thing to model. A species grows when there is plenty of resources. It falls when resources are exhausted and/or too much waste is generated. So it would seem that if there are very few of a species, then there are plenty of resources and not much waste, and the population should grow. However(!), there is an additional way that it can fall: the population can fall because there are so few of a species that they cannot find mates. For example, what if there is only one mountain lion in our entire valley. Even if it has plenty of rabbits to eat, it may not find a mate. This is called the Allee effect.

There is a differential equation for population growth that takes this into account:

$$P'(t) \equiv \frac{dP}{dt} \equiv \underset{\text{as teeny as }\Delta t\text{ can be made}}{\text{limit}} \frac{\Delta P}{\Delta t} = rP\left(\frac{P}{a} - 1\right)\left(1 - \frac{P}{k}\right)$$

In this equation, r, a, and k are just numbers. The number k has the interpretation of being the stable population amount. The number a has the interpretation of being the population amount below which mates get too hard to find. The number r just sets the time scale: do booms happen on the scale of hours (as in mold on warm food), or decades (as in humans on new terrain)?

We are going to choose $r = 1$, $a = 2$, and $k = 5$. So the equation is:

$$P'(t) = P\left(\frac{P}{2} - 1\right)\left(1 - \frac{P}{5}\right)$$

(a) Apply the differential equations solver to this equation with $P(0) = 1$, where we expect collapse.

(b) Apply it again with $P(0) = 3$, where we expect growth toward equilibrium.

(c) Apply it one more time with $P(0) = 10$, where we expect a decline toward equilibrium.

Make tables and graphs for all three cases.

2. Eutrophication

In agricultural areas, humans often put a lot of fertilizer on their crops, especially phosphates, and the excess runs off into streams and then into lakes. It is pretty common to see the lakes covered with a thick mat of algae. This isn't a big surprise. The algae on the surface gets both fertilizer and sunlight and grows fast. The algae at the bottom of the mat rots, and rotting consumes oxygen, and the animal life underneath the algae bloom suffocates. (I am not an ecologist, but this is the basic idea. Feel free to read up on eutrophication if you want more detail and accuracy.)

Eutrophication can be modeled by focusing on the phosphorus concentration:

$$P'(t) \equiv \frac{dP}{dt} \equiv \underset{\text{as teeny as } \Delta t \text{ can be made}}{\text{limit}} \frac{\Delta P}{\Delta t} = I - sP + r\,\frac{P^n}{M^n + P^n}$$

I, s, M, and n are just numbers that ecologists put into the model. I is the input rate of the phosphorus into the body of water. s sets the rate of outflow and sedimentation.

The last term is the one that is responsible for the algae explosion if it dominates. The interpretation is that it represents the spread into the water of the phosphorus from the sediment. If $P \ll M$, the last term is negligible. If $P \gg M$ the last term simplifies to r. Don't take my word for it. Definitely contemplate those two cases and understand the behavior of the last term.

A simplified version of this model has $s = 1$ and $M = 1$. This simplification actually just corresponds to changing the units for time and the units for concentration. So we are down to:

$$P'(t) = I - P + r\,\frac{P^n}{1 + P^n}$$

r and n are the next things that have to be chosen. Take $r = 5$ and $n = 8$. (Don't ask me for the experimental justification for those choices. From here on, I am just making an example that I don't personally have any particular understanding of even though up to this point, I understood, and hopefully made understandable to you, the ecological model.)

With these choices for r and n, we have:

$$P'(t) = I - P + 5\,\frac{P^8}{1 + P^8}$$

The last thing to choose is I. The interesting thing about this differential equation is that it has a completely different behavior for $I = 1/4$ vs. $I = 3/4$.

In other words, a small change in the input rate can result in a wild change in how much algae bloom forms, and this is exactly what happens in the real world: a lake seems fine until the conditions change a little (perhaps a modest decrease in rainfall or a modest increase in fertilizer, either of which have the effect of changing the concentration of the phosphorus in the input), and then it suddenly eutrophies.

We would like to see if we can see this sensitive effect.

(a) Apply the differential equations solver to this equation with $P(0) = 1/2$, and $I = 1/4$. With some luck, we will see that the phosphorus settles out into the sediment.

(b) Apply it again with $P(0) = 1/2$, and $I = 3/4$. We should see that the lake eutrophies.

As usual, make tables and graphs for the cases.