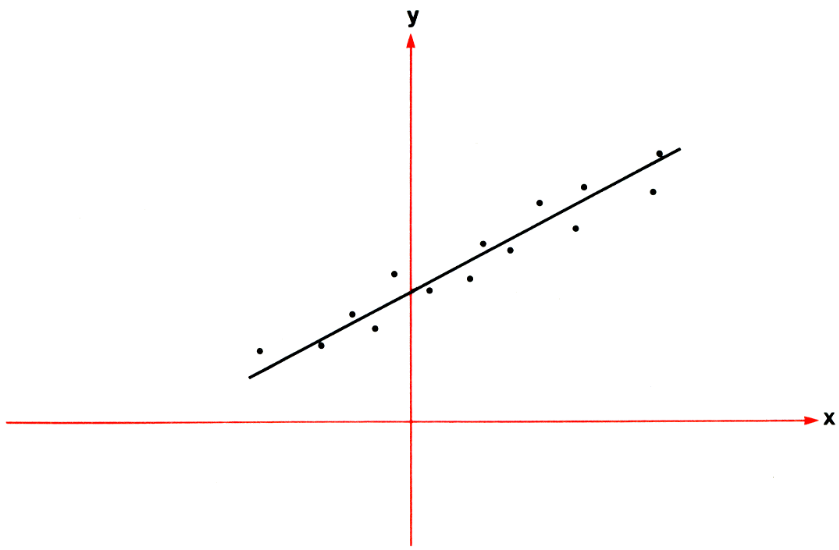
CHAPTER 6 STATISTICS

CURVE FITTING—LINEAR REGRESSION



When investigating the relationship between two variables in the real world, it is a reasonable first step to make experimental observations of the system to gather paired values of the variables, (x, y). The investigator might then ask the question: What mathematical formula best describes the relationship between the variables x and y? His first guess will often be that the relationship is linear, i.e., that the form of the equation is $y = a_1 x + a_0$, where a_1 and a_0 are constants. The purpose of this program is to find the constants a_1 and a_0, which give the closest agreement between the experimental data and the equation $y = a_1 x + a_0$. The technique used is linear regression by the method of least squares.

The user must input the paired values of data he has gathered, (x_i, y_i), $i = 1, ..., n$. When all data pairs have been input, the regression constants a_1 and a_0 may be calculated. A third value may also be found, the coefficient of determination, r^2. The value of r^2 will lie between 0 and 1 and will indicate how closely the equation fits the experimental data: the closer r^2 is to 1, the better the fit.

Equations:

$$y = a_1 x + a_0$$

All summations below are performed for $i = 1, ..., n$.

Regression constants:

$$a_1 = \frac{\Sigma xy - \dfrac{\Sigma x \, \Sigma y}{n}}{\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}}$$

$$a_0 = \bar{y} - a_1 \bar{x}$$

where $\bar{y} = \dfrac{\Sigma y}{n}$

$\bar{x} = \dfrac{\Sigma x}{n}$

Coefficient of determination:

$$r^2 = \frac{\left[\Sigma xy - \dfrac{\Sigma x \Sigma y}{n}\right]^2}{\left[\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}\right]\left[\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}\right]}$$

Note:

The values for a_0 and a_1 are stored in R_0 and R_1, respectively. After the calculation of a_0, a_1, and r^2, the estimated y-value, $\hat{y}$, corresponding to any x-value may be calculated by $y = a_1 x + a_0$.

Programming Remarks:

The intermediate value $C = \Sigma xy - (\Sigma x \Sigma y/n)$ is first calculated at line 14 but is also needed near the end of the program to find r^2. Since all registers R_0 through R_7 are in use, the only place to save this value is in the stack. Hence C is preserved in one or more of the stack registers from lines 14 through 36, when it is used. It is due to the presence of C in the stack that users are warned not to disturb the contents of the stack after calculation of a_0 and a_1 (see step 4 of User Instructions).

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS
LINE	CODE						
00			y	x			Steps 1-7 for summation
01	31	↑	y	y	x		
02	15 02	g x^2	y^2	y	x		
03	23 51 02	STO + 2	y^2	y	x		Σy^2
04	22	R↓	y	x		y^2	
05	21	x⇄y	x	y		y^2	
06	25	Σ+	n	y		y^2	n, Σy, Σxy, Σx^2, Σx
07	13 00	GTO 00	n	y		y^2	
08	24 05	RCL 5	Σxy				
09	24 07	RCL 7	Σx	Σxy			
10	24 04	RCL 4	Σy	Σx	Σxy		
11	61	x	$\Sigma x\ \Sigma y$	Σxy			
12	24 03	RCL 3	n	$\Sigma x\ \Sigma y$	Σxy		
13	71	÷	$\Sigma x\ \Sigma y/n$	Σxy			
14	41	−	C				$C = \Sigma xy - (\Sigma x\ \Sigma y/n)$
15	24 06	RCL 6	Σx^2	C			
16	24 07	RCL 7	Σx	Σx^2	C		
17	15 02	g x^2	$(\Sigma x)^2$	Σx^2	C		
18	24 03	RCL 3	n	$(\Sigma x)^2$	Σx^2	C	
19	71	÷	$(\Sigma x)^2/n$	Σx^2	C	C	
20	41	−	D	C	C	C	$D = \Sigma x^2 - [(\Sigma x)^2/n]$
21	71	÷	a_1	C	C	C	$a_1 = C/D$
22	23 01	STO 1	a_1	C	C	C	
23	24 07	RCL 7	Σx	a_1	C	C	
24	61	x	$a_1\ \Sigma x$	C	C	C	
25	32	CHS	$-a_1\ \Sigma x$	C	C	C	
26	24 04	RCL 4	Σy	$-a_1\ \Sigma x$	C	C	
27	51	+	$\Sigma y - a_1\ \Sigma x$	C	C	C	
28	24 03	RCL 3	n	$\Sigma y - a_1\ \Sigma x$	C	C	
29	71	÷	a_0	C	C	C	$a_0 = \bar{y} - a_1\ \bar{x}$
30	23 00	STO 0	a_0	C	C	C	
31	74	R/S	a_0	C	C	C	Halt to display a_0
32	24 01	RCL 1	a_1	a_0	C	C	
33	74	R/S	a_1	a_0	C	C	Halt to display a_1
34	21	x⇄y	a_0	a_1	C	C	
35	22	R↓	a_1	C	C	a_0	
36	61	x	a_1 C	C	a_0	a_0	
37	24 02	RCL 2	Σy^2	a_1 C	C	a_0	
38	24 04	RCL 4	Σy	Σy^2	a_1 C	C	
39	15 02	g x^2	$(\Sigma y)^2$	Σy^2	a_1 C	C	
40	24 03	RCL 3	n	$(\Sigma y)^2$	Σy^2	a_1 C	
41	71	÷	$(\Sigma y)^2/n$	Σy^2	a_1 C	a_1 C	
42	41	−	E	a_1 C	a_1 C	a_1 C	$E = \Sigma y^2 - [(\Sigma y)^2/n]$
43	71	÷	r^2	a_1 C	a_1 C	a_1 C	$r^2 = a_1\ C/E$
44	13 00	GTO 00	r^2	a_1 C	a_1 C	a_1 C	
45							
46							
47							
48							
49							

REGISTERS	
R_0	a_0
R_1	a_1
R_2	Σy^2
R_3	n
R_4	Σy
R_5	Σxy
R_6	Σx^2
R_7	Σx

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Perform for i = 1,..., n:						
	Input x-value and y-value	x_i	↑				
		y_i	R/S				i
4	Compute regression constants		GTO	08	R/S		a_0 *
			R/S				a_1 *
5	Compute coefficient of deter-						
	mination		R/S				r^2
6	To calculate a projected y-value,						
	input the x-value	x	RCL	1	×	RCL	
			0	+			$\hat{y}$
7	Perform step 6 as many times as						
	desired						
8	For a new case, go to step 2.						
	* The contents of the stack						
	should not be disturbed at these						
	points.						

Example:

An eccentric professor of numerical analysis wakes up one morning and feels feverish. A search through his medicine cabinet reveals one oral thermometer which, unfortunately, is in degrees centigrade, a scale he is not familiar with. As he stares disconsolately out his window, he spies the outdoor thermometer affixed to the windowframe. This thermometer, however, will not fit comfortably into his mouth. Still, with some ingenuity....

The professor suspects that the relationship is $F = a_1 C + a_0$. If he can get a few data pairs for F and C, he can run a linear regression program to find a_1 and a_0, then convert any reading in °C to °F through the equation. So tossing both thermometers into a sink of lukewarm water, he reads the following pairs of temperatures as the water cools:

C	40.5	38.6	37.9	36.2	35.1	34.6
F	104.5	102	100	97.5	95.5	94

If the relationship is indeed $F = a_1 C + a_0$, what are the values for a_1 and a_0? What is the coefficient of determination?

Solution:

[f] [PRGM] [f] [REG] 40.5 [↑] 104.5 [R/S] ⟶ 1.00

38.6 [↑] 102 [R/S] ⟶ 2.00

37.9 [↑] 100 [R/S] ⟶ 3.00

36.2 [↑] 97.5 [R/S] ⟶ 4.00

35.1 [↑] 95.5 [R/S] ⟶ 5.00

34.6 [↑] 94 [R/S] ⟶ 6.00

[GTO] [0] [8] [R/S] ⟶ 33.53

[R/S] ⟶ 1.76

[R/S] ⟶ 0.99

Thus, by the data above, F = 1.76 C + 33.53, with $r^2 = 0.99$. (The real equation, of course, is F = 1.8C + 32.)

Suppose the professor puts the centigrade thermometer in his mouth and finds he has a temperature of 37°C. Should he be worried?

37 [RCL] [1] [×] [RCL] [0] [+] ⟶ 98.65°F

It looks like he is safe.

EXPONENTIAL CURVE FIT

This program computes the least squares fit of n pairs of data points $\{(x_i, y_i), i = 1, 2, ..., n\}$, where $y_i > 0$, for an exponential function of the form

$$y = a e^{bx} \qquad (a > 0).$$

The equation is linearized into

$$\ln y = \ln a + bx.$$

The following statistics are computed:

1. Coefficients a, b

$$b = \frac{\Sigma x_i \ln y_i - \frac{1}{n}(\Sigma x_i)(\Sigma \ln y_i)}{\Sigma x_i^2 - \frac{1}{n}(\Sigma x_i)^2}$$

$$a = \exp\left[\frac{\Sigma \ln y_i}{n} - b\frac{\Sigma x_i}{n}\right]$$

2. Coefficient of determination

$$r^2 = \frac{\left[\Sigma x_i \ln y_i - \frac{1}{n}\Sigma x_i \Sigma \ln y_i\right]^2}{\left[\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}\right]\left[\Sigma(\ln y_i)^2 - \frac{(\Sigma \ln y_i)^2}{n}\right]}$$

3. Estimated value $\hat{y}$ for a given x

$$\hat{y} = a e^{bx}$$

Note:

n is a positive integer and $n \neq 1$.

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
00		
01	14 07	f LN
02	31	↑
03	15 02	g x^2
04	23 51 02	STO + 2
05	22	R↓
06	21	x⇄y
07	25	Σ+
08	13 00	GTO 00
09	24 05	RCL 5
10	24 07	RCL 7
11	24 04	RCL 4
12	61	x
13	24 03	RCL 3
14	71	÷
15	41	–
16	24 06	RCL 6
17	24 07	RCL 7
18	15 02	g x^2
19	24 03	RCL 3
20	71	÷
21	41	–
22	71	÷
23	23 01	STO 1
24	24 07	RCL 7

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
25	61	x
26	32	CHS
27	24 04	RCL 4
28	51	+
29	24 03	RCL 3
30	71	÷
31	15 07	g e^x
32	23 00	STO 0
33	74	R/S
34	24 01	RCL 1
35	74	R/S
36	21	x⇄y
37	22	R↓
38	61	x
39	24 02	RCL 2
40	24 04	RCL 4
41	15 02	g x^2
42	24 03	RCL 3
43	71	÷
44	41	–
45	71	÷
46	13 00	GTO 00
47		
48		
49		

REGISTERS
R_0 a
R_1 b
R_2 $\Sigma (\ln y)^2$
R_3 n
R_4 $\Sigma \ln y$
R_5 $\Sigma x \ln y$
R_6 Σx^2
R_7 Σx

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Perform for i = 1,..., n:						
	Input x-value and y-value	x_i	↑				
		y_i	R/S				i
4	Compute constants		GTO	09	R/S		a*
			R/S				b*
5	Compute coefficient of deter-						
	mination		R/S				r^2
6	To calculate $\hat{y}$, input x	x	RCL	1	x	g	
			e^x	RCL	0	x	$\hat{y}$
7	Perform step 6 as many times						
	as desired						
8	For new case, go to step 2.						
	* The stack must be maintained						
	at these points.						

Example:

x_i	.72	1.31	1.95	2.58	3.14
y_i	2.16	1.61	1.16	.85	0.5

Solution:

$a = 3.45$, $b = -0.58$

$y = 3.45\, e^{-0.58x}$

$r^2 = 0.98$

For $x = 1.5$, $\hat{y} = 1.44$

LOGARITHMIC CURVE FIT

This program fits a logarithmic curve

$$y = a + b \ln x$$

to a set of data points

$$\{(x_i, y_i), i = 1, 2, ..., n\}$$

where $x_i > 0$.

Program computes:

1. Regression coefficients

$$b = \frac{\Sigma y_i \ln x_i - \frac{1}{n} \Sigma \ln x_i \Sigma y_i}{\Sigma (\ln x_i)^2 - \frac{1}{n} (\Sigma \ln x_i)^2}$$

$$a = \frac{1}{n} (\Sigma y_i - b \Sigma \ln x_i)$$

2. Coefficient of determination

$$r^2 = \frac{\left[\Sigma y_i \ln x_i - \frac{1}{n} \Sigma \ln x_i \Sigma y_i\right]^2}{\left[\Sigma (\ln x_i)^2 - \frac{1}{n} (\Sigma \ln x_i)^2\right] \left[\Sigma y_i^2 - \frac{1}{n} (\Sigma y_i)^2\right]}$$

3. Estimated value $\hat{y}$ for given x

$$\hat{y} = a + b \ln x$$

Note:

n is a positive integer and $n \neq 1$.

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
00		
01	31	↑
02	15 02	g x^2
03	23 51 02	STO + 2
04	22	R↓
05	21	x⇄y
06	14 07	f LN
07	25	Σ+
08	13 00	GTO 00
09	24 05	RCL 5
10	24 07	RCL 7
11	24 04	RCL 4
12	61	x
13	24 03	RCL 3
14	71	÷
15	41	–
16	24 06	RCL 6
17	24 07	RCL 7
18	15 02	g x^2
19	24 03	RCL 3
20	71	÷
21	41	–
22	71	÷
23	23 01	STO 1
24	24 07	RCL 7
25	61	x
26	32	CHS
27	24 04	RCL 4
28	51	+
29	24 03	RCL 3
30	71	÷
31	23 00	STO 0
32	74	R/S
33	24 01	RCL 1
34	74	R/S
35	21	x⇄y
36	22	R↓
37	61	x
38	24 02	RCL 2
39	24 04	RCL 4
40	15 02	g x^2
41	24 03	RCL 3
42	71	÷
43	41	–
44	71	÷
45	13 00	GTO 00
46		
47		
48		
49		

REGISTERS
R_0 a
R_1 b
R_2 Σy^2
R_3 n
R_4 Σy
R_5 $\Sigma y \ln x$
R_6 $\Sigma \ln x$
R_7 $\Sigma (\ln x)^2$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Perform for i = 1,..., n:						
	Input x-value and y-value	x_i	↑				
		y_i	R/S				i
4	Compute constants		GTO	09	R/S		a*
			R/S				b*
5	Compute coefficient of determination						
			R/S				r^2
6	To calculate $\hat{y}$, input x	x	f	ln	RCL	1	
			x	RCL	0	+	$\hat{y}$
7	Perform step 6 as many times as desired						
8	For new case, go to step 2.						
	* The stack must be maintained at these points						

Example:

x_i	3	4	6	10	12
y_i	1.5	9.3	23.4	45.8	60.1

Solution:

$a = -47.02$, $b = 41.39$

$y = -47.02 + 41.39 \ln x$

$r^2 = 0.98$

For $x = 8$, $\hat{y} = 39.06$

For $x = 14.5$, $\hat{y} = 63.67$

POWER CURVE FIT

This program fits a power curve

$$y = ax^b \qquad (a > 0)$$

to a set of data points

$$\{(x_i, y_i), i = 1, 2, ..., n\}$$

where $x_i > 0, y_i > 0$.

By writing this equation as

$$\ln y = b \ln x + \ln a$$

the problem can be solved as a linear regression problem.

Output statistics are:

1. Regression coefficients

$$b = \frac{\Sigma (\ln x_i)(\ln y_i) - \frac{(\Sigma \ln x_i)(\Sigma \ln y_i)}{n}}{\Sigma (\ln x_i)^2 - \frac{(\Sigma \ln x_i)^2}{n}}$$

$$a = \exp\left[\frac{\Sigma \ln y_i}{n} - b\frac{\Sigma \ln x_i}{n}\right]$$

2. Coefficient of determination

$$r^2 = \frac{\left[\Sigma (\ln x_i)(\ln y_i) - \frac{(\Sigma \ln x_i)(\Sigma \ln y_i)}{n}\right]^2}{\left[\Sigma (\ln x_i)^2 - \frac{(\Sigma \ln x_i)^2}{n}\right]\left[\Sigma (\ln y_i)^2 - \frac{(\Sigma \ln y_i)^2}{n}\right]}$$

3. Estimated value $\hat{y}$ for given x

$$\hat{y} = ax^b$$

Note:

n is a positive integer and $n \neq 1$.

LINE	CODE	KEY ENTRY
00		
01	14 07	f LN
02	31	↑
03	15 02	g x^2
04	23 51 02	STO + 2
05	22	R↓
06	21	x⇄y
07	14 07	f LN
08	25	Σ+
09	13 00	GTO 00
10	24 05	RCL 5
11	24 07	RCL 7
12	24 04	RCL 4
13	61	x
14	24 03	RCL 3
15	71	÷
16	41	–
17	24 06	RCL 6
18	24 07	RCL 7
19	15 02	g x^2
20	24 03	RCL 3
21	71	÷
22	41	–
23	71	÷
24	23 01	STO 1

LINE	CODE	KEY ENTRY
25	24 07	RCL 7
26	61	x
27	32	CHS
28	24 04	RCL 4
29	51	+
30	24 03	RCL 3
31	71	÷
32	15 07	g e^x
33	23 00	STO 0
34	74	R/S
35	24 01	RCL 1
36	74	R/S
37	21	x⇄y
38	22	R↓
39	61	x
40	24 02	RCL 2
41	24 04	RCL 4
42	15 02	g x^2
43	24 03	RCL 3
44	71	÷
45	41	–
46	71	÷
47	13 00	GTO 00
48		
49		

REGISTERS
R_0 a
R_1 b
R_2 $\Sigma (\ln y)^2$
R_3 n
R_4 $\Sigma \ln y$
R_5 $\Sigma (\ln x)(\ln y)$
R_6 $\Sigma (\ln x)^2$
R_7 $\Sigma \ln x$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Perform for i = 1,..., n:						
	Input x-value and y-value	x_i	↑				
		y_i	R/S				i
4	Compute constants		GTO	10	R/S		a*
			R/S				b*
5	Compute coefficient of deter-						
	mination		R/S				r^2
6	Input x-value and compute $\hat{y}$	x	RCL	1	f	y^x	
			RCL	0	x		$\hat{y}$
7	Perform step 6 as many times as						
	desired						
8	For new case, go to step 2.						
	* The stack must be maintained						
	at these points.						

Example:

x_i	10	12	15	17	20	22	25	27	30	32	35
y_i	0.95	1.05	1.25	1.41	1.73	2.00	2.53	2.98	3.85	4.59	6.02

Solution:

$a = .03,\ b = 1.46$

$y = .03x^{1.46}$

$r^2 = 0.94$

For $x = 18,\ \hat{y} = 1.76$

$x = 23,\ \hat{y} = 2.52$

COVARIANCE AND CORRELATION COEFFICIENT

For a set of given data points $\{(x_i, y_i), i = 1, 2, ..., n\}$, the covariance and the correlation coefficent are defined as:

$$\text{covariance } s_{xy} = \frac{1}{n-1}\left(\Sigma x_i y_i - \frac{1}{n}\Sigma x_i \Sigma y_i\right)$$

$$\text{or } s_{xy}' = \frac{1}{n}\left(\Sigma x_i y_i - \frac{1}{n}\Sigma x_i \Sigma y_i\right)$$

$$\text{correlation coefficient } r = \frac{s_{xy}}{s_x s_y}$$

where s_x and s_y are standard deviations

$$s_x = \sqrt{\frac{\Sigma x_i^2 - (\Sigma x_i)^2/n}{n-1}} \qquad s_y = \sqrt{\frac{\Sigma y_i^2 - (\Sigma y_i)^2/n}{n-1}}$$

Note:

$-1 \leqslant r \leqslant 1$

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
00		
01	31	↑
02	15 02	g x^2
03	23 51 02	STO + 2
04	22	R↓
05	21	x⇄y
06	25	Σ+
07	13 00	GTO 00
08	24 05	RCL 5
09	24 04	RCL 4
10	24 07	RCL 7
11	61	x
12	24 03	RCL 3
13	71	÷
14	41	–
15	24 03	RCL 3
16	01	1
17	41	–
18	23 00	STO 0
19	71	÷
20	23 01	STO 1
21	74	R/S
22	24 00	RCL 0
23	61	x
24	24 03	RCL 3

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
25	71	÷
26	74	R/S
27	14 22	f s
28	23 71 01	STO ÷ 1
29	24 02	RCL 2
30	24 04	RCL 4
31	15 02	g x^2
32	24 03	RCL 3
33	71	÷
34	41	–
35	24 00	RCL 0
36	71	÷
37	14 02	f $\sqrt{x}$
38	23 71 01	STO ÷ 1
39	24 01	RCL 1
40	13 00	GTO 00
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R_0 n – 1
R_1 Used
R_2 Σy^2
R_3 n
R_4 Σy
R_5 Σxy
R_6 Σx^2
R_7 Σx

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM	f	REG	
3	Perform this step for i = 1, 2,...,n	x_i	↑				
		y_i	R/S				i
4	Compute covariance s_{xy}		GTO	08	R/S		s_{xy}
5	Compute s_{xy}'		R/S				s_{xy}'
6	Compute correlation coefficient		R/S				r
7	For new case, go to step 2.						

Example:

x_i	26	30	44	50	62	68	74
y_i	92	85	78	81	54	51	40

Solution:

$s_{xy} = -354.14$

$s_{xy}' = -303.55$

$r = -0.96$

MOMENTS AND SKEWNESS

This program computes the following statistics for a set of given data $\{x_1, x_2, ..., x_n\}$:

1st moment $\quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

2nd moment $\quad m_2 = \frac{1}{n}\Sigma x_i^2 - \bar{x}^2$

3rd moment $\quad m_3 = \frac{1}{n}\Sigma x_i^3 - \frac{3}{n}\bar{x}\,\Sigma x_i^2 + 2\bar{x}^3$

moment coefficient of skewness

$$\gamma_1 = \frac{m_3}{m_2^{3/2}}$$

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
00		
01	31	↑
02	15 02	g x^2
03	25	Σ+
04	13 00	GTO 00
05	24 04	RCL 4
06	24 03	RCL 3
07	71	÷
08	23 02	STO 2
09	74	R/S
10	24 07	RCL 7
11	24 03	RCL 3
12	71	÷
13	24 02	RCL 2
14	15 02	g x^2
15	41	–
16	23 01	STO 1
17	74	R/S
18	24 05	RCL 5
19	24 03	RCL 3
20	71	÷
21	24 07	RCL 7
22	24 02	RCL 2
23	61	x
24	24 03	RCL 3

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
25	71	÷
26	03	3
27	61	x
28	41	–
29	24 02	RCL 2
30	31	↑
31	15 02	g x^2
32	61	x
33	02	2
34	61	x
35	51	+
36	23 00	STO 0
37	74	R/S
38	24 00	RCL 0
39	24 01	RCL 1
40	01	1
41	73	•
42	05	5
43	14 03	f y^x
44	71	÷
45	13 00	GTO 00
46		
47		
48		
49		

REGISTERS
R_0 m_3
R_1 m_2
R_2 $\bar{x}$
R_3 n
R_4 Σx
R_5 Σx^3
R_6 Σx^4
R_7 Σx^2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM	f	REG	
3	Perform for i = 1, 2,..., n:						
	Input x-value	x_i	R/S				i
4	Delete erroneous data	x_k	↑	g	x^2	f	
			Σ–				
5	Compute the mean		GTO	05	R/S		$\bar{x}$
6	Compute the second and third						
	moments		R/S				m_2
			R/S				m_3
7	Compute the moment coefficient						
	of skewness		R/S				γ_1
8	For new case, go to step 2.						

Example:

i	1	2	3	4	5	6	7	8	9
x_i	2.1	3.5	4.2	6.5	4.1	3.6	5.3	3.7	4.9

Solution:

$\bar{x} = 4.21$

$m_2 = 1.39$

$m_3 = 0.39$

$\gamma_1 = 0.24$

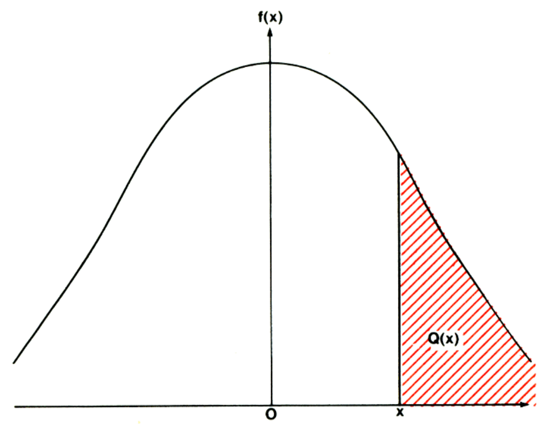
NORMAL DISTRIBUTION

The density function for a standard normal variable is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} .$$

The upper tail area is

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt .$$



For $x \geqslant 0$, polynomial approximation is used to compute Q(x):

$$Q(x) = f(x)\,(b_1\, t + b_2\, t^2 + b_3\, t^3 + b_4\, t^4 + b_5\, t^5) + \epsilon(x)$$

where $|\epsilon(x)| < 7.5 \times 10^{-8}$

$$t = \frac{1}{1 + rx}, \quad r = 0.2316419$$

$b_1 = .31938153$, $b_2 = -.356563782$

$b_3 = 1.781477937$, $b_4 = -1.821255978$

$b_5 = 1.330274429$

Note:

The program only works for $x \geqslant 0$. Equations $f(-x) = f(x)$, $Q(-x) = 1-Q(x)$, where $x \geqslant 0$, can be used to find f and Q for negative numbers.

Reference:

Abramowitz and Stegun, *Handbook of Mathematical Functions,* National Bureau of Standards, 1968.

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
00		
01	31	↑
02	23 06	STO 6
03	61	x
04	02	2
05	71	÷
06	32	CHS
07	15 07	g e^x
08	15 73	g π
09	02	2
10	61	x
11	14 02	f $\sqrt{x}$
12	71	÷
13	23 07	STO 7
14	74	R/S
15	24 00	RCL 0
16	24 06	RCL 6
17	61	x
18	01	1
19	51	+
20	15 22	g 1/x
21	31	↑
22	31	↑
23	31	↑
24	24 05	RCL 5
25	61	x
26	24 04	RCL 4
27	51	+
28	61	x
29	24 03	RCL 3
30	51	+
31	61	x
32	24 02	RCL 2
33	51	+
34	61	x
35	24 01	RCL 1
36	51	+
37	61	x
38	24 07	RCL 7
39	61	x
40	13 00	GTO 00
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R_0	r
R_1	b_1
R_2	b_2
R_3	b_3
R_4	b_4
R_5	b_5
R_6	x
R_7	f(x)

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM			
3	Store constants	r	STO	0			
		b_1	STO	1			
		b_2	STO	2			
		b_3	STO	3			
		b_4	STO	4			
		b_5	STO	5			
4	Input x and compute f(x)	x	R/S				f(x)
5	Compute Q(x)		R/S				Q(x)
6	For a new case, go to 4.						

Examples:

1. $x = 1.18$

2. $x = 2.28$

Solutions:

1. $f(x) = 0.20$

   $Q(x) = 0.12$

2. $f(x) = 0.03$

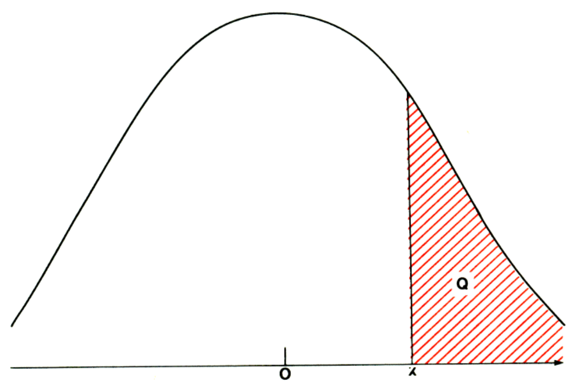
   $Q(x) = 0.01$

INVERSE NORMAL INTEGRAL

This program determines the value of x such that

$$Q = \int_x^\infty \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\,dt$$

where Q is given and $0 < Q \leqslant 0.5$.



The following rational approximation is used:

$$x = t - \frac{c_0 + c_1 t + c_2 t^2}{1 + d_1 t + d_2 t^2 + d_3 t^3} + \epsilon(Q)$$

where $|\epsilon(Q)| < 4.5 \times 10^{-4}$

$$t = \sqrt{\ln \frac{1}{Q^2}}$$

$c_0 = 2.515517$ $d_1 = 1.432788$

$c_1 = 0.802853$ $d_2 = 0.189269$

$c_2 = 0.010328$ $d_3 = 0.001308$

Reference:

Abramowitz and Stegun, *Handbook of Mathematical Functions,* National Bureau of Standards, 1968.

DISPLAY LINE	CODE	KEY ENTRY
00		
01	31	↑
02	61	x
03	15 22	g 1/x
04	14 07	f LN
05	14 02	f $\sqrt{x}$
06	23 06	STO 6
07	31	↑
08	31	↑
09	31	↑
10	24 05	RCL 5
11	61	x
12	24 04	RCL 4
13	51	+
14	61	x
15	24 03	RCL 3
16	51	+
17	61	x
18	01	1
19	51	+
20	23 07	STO 7
21	34	CLX
22	24 02	RCL 2
23	61	x
24	24 01	RCL 1
25	51	+
26	61	x
27	24 00	RCL 0
28	51	+
29	24 07	RCL 7
30	71	÷
31	41	–
32	13 00	GTO 00
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R_0	c_0
R_1	c_1
R_2	c_2
R_3	d_1
R_4	d_2
R_5	d_3
R_6	t
R_7	$1 + d_1 t + d_2 t^2 + d_3 t^3$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM			
3	Store constants	c_0	STO	0			
		c_1	STO	1			
		c_2	STO	2			
		d_1	STO	3			
		d_2	STO	4			
		d_3	STO	5			
4	Input Q	Q	R/S				
5	For a new case, go to 4.						

Examples:

1. Q = 0.12
2. Q = 0.05

Solutions:

1. x = 1.18
2. x = 1.65

FACTORIAL

This program will compute factorials for positive integers between 2 and 69.

$$n! = n(n-1)(n-2)\ldots(2)(1)$$

Notes:

1. For large values of n, the program will take some time to arrive at a result, up to a maximum of about 20 seconds for n = 69.

2. The program does not check input values and will return incorrect answers for values of n < 2 or n > 69 or n non-integer.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	31	↑
02	01	1
03	23 00	STO 0
04	21	x⇄y
05	23 61 00	STO × 0
06	01	1
07	41	–
08	14 61	f x≠y
09	13 05	GTO 05
10	24 00	RCL 0
11	13 00	GTO 00
12		
13		
14		
15		
16		
17		
18		
19		
20		
21		
22		
23		
24		

DISPLAY		KEY ENTRY
LINE	CODE	
25		
26		
27		
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R_0 Used
R_1
R_2
R_3
R_4
R_5
R_6
R_7

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM			
3	Key in n ($2 \leqslant n \leqslant 69$)	n	R/S				n!
4	For a new n, go to step 3.						

Examples:

1. 5! = 120.00

2. 10! = 3628800.00

PERMUTATION

A permutation is an ordered subset of a set of distinct objects. The number of possible permutations, each containing n objects, that can be formed from a collection of m distinct objects is given by

$$_mP_n = \frac{m!}{(m-n)!} = m(m-1)\ldots(m-n+1)$$

where m, n are integers and $0 \leqslant n \leqslant m$.

Notes:

1. $_mP_n$ can also be denoted by P_n^m, $P(m,n)$ or $(m)_n$.
2. $_mP_0 = 1$, $_mP_1 = m$, $_mP_m = m!$

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
00		
01	24 00	RCL 0
02	24 00	RCL 0
03	24 01	RCL 1
04	15 71	g x=0
05	13 29	GTO 29
06	14 71	f x=y
07	13 31	GTO 31
08	14 51	f x⩾y
09	13 39	GTO 39
10	01	1
11	14 71	f x=y
12	13 41	GTO 41
13	22	R↓
14	41	−
15	01	1
16	51	+
17	61	x
18	14 73	f LASTx
19	24 00	RCL 0
20	01	1
21	41	−
22	14 71	f x=y
23	13 26	GTO 26
24	22	R↓

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
25	13 15	GTO 15
26	22	R↓
27	22	R↓
28	13 00	GTO 00
29	01	1
30	13 00	GTO 00
31	01	1
32	41	−
33	15 71	g x=0
34	13 37	GTO 37
35	23 61 00	STO x 0
36	13 31	GTO 31
37	24 00	RCL 0
38	13 00	GTO 00
39	00	0
40	71	÷
41	22	R↓
42	22	R↓
43	13 00	GTO 00
44		
45		
46		
47		
48		
49		

REGISTERS
R_0 m
R_1 n
R_2
R_3
R_4
R_5
R_6
R_7

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store m, n	m	STO	0			
		n	STO	1			
3	Compute permutations		f	PRGM	R/S		$_mP_n$
4	For new case, go to step 2.						

Examples:

1. $_{43}P_3$ = 74046.00

2. $_{73}P_4$ = 26122320.00

COMBINATION

A combination is a selection of one or more of a set of distinct objects without regard to order. The number of possible combinations, each containing n objects, that can be formed from a collection of m distinct objects is given by

$$ {}_mC_n = \frac{m!}{(m-n)!\, n!} = \frac{m(m-1)\ldots(m-n+1)}{1 \cdot 2 \cdot \ldots \cdot n} $$

where m, n are integers and $0 \leqslant n \leqslant m$.

This program computes ${}_mC_n$ using the following algorithm:

1. If $n \leqslant m - n$

$$ {}_mC_n = \frac{m-n+1}{1} \cdot \frac{m-n+2}{2} \cdot \ldots \cdot \frac{m}{n} . $$

2. If $n > m - n$, program computes ${}_mC_{m-n}$.

Notes:

1. ${}_mC_n$, which is also called the binomial coefficient, can be denoted by C_n^m, C(m,n), or $\binom{m}{n}$.
2. ${}_mC_n = {}_mC_{m-n}$
3. ${}_mC_0 = {}_mC_m = 1$
4. ${}_mC_1 = {}_mC_{m-1} = m$

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
00		
01	41	−
02	14 73	f LASTx
03	14 41	f x<y
04	21	x⇄y
05	23 00	STO 0
06	01	1
07	23 01	STO 1
08	51	+
09	23 02	STO 2
10	22	R↓
11	15 71	g x=0
12	13 30	GTO 30
13	01	1
14	24 01	RCL 1
15	51	+
16	23 01	STO 1
17	21	x⇄y
18	14 51	f x≥y
19	13 22	GTO 22
20	24 02	RCL 2
21	13 00	GTO 00
22	22	x⇄y
23	24 00	RCL 0
24	51	+
25	24 01	RCL 1
26	71	÷
27	23 61 02	STO x 2
28	22	R↓
29	13 13	GTO 13
30	01	1
31	13 00	GTO 00
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R_0 max (n, m − n)
R_1 Used
R_2 Used
R_3
R_4
R_5
R_6
R_7

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Key in m and n	m	↑				
		n	f	PRGM	R/S		${}_mC_n$
3	For new case, go to step 2.						

Examples:

1. ${}_{73}C_4 = 1088430.00$

2. ${}_{43}C_3 = 12341.00$

RANDOM NUMBER GENERATOR

This program calculates uniformly distributed pseudo random numbers u_i in the range

$$0 \leqslant u_i \leqslant 1$$

using the following formula:

$$u_i = \text{Fractional part of } [(\pi + u_{i-1})^5].$$

The user has to specify the starting value u_0 (the "seed" of the sequence) such that

$$0 \leqslant u_0 \leqslant 1.$$

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
00		
01	15 73	g π
02	24 00	RCL 0
03	51	+
04	05	5
05	14 03	f y^x
06	15 01	g FRAC
07	23 00	STO 0
08	13 00	GTO 00
09		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20		
21		
22		
23		
24		

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
25		
26		
27		
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R_0 u_i
R_1
R_2
R_3
R_4
R_5
R_6
R_7

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store seed	u_0	STO	0	f	PRGM	
3	Generate random number		R/S				u_i
4	Repeat step 3 as many times as						
	desired						
5	For new sequence, go to step 2.						

Example:

Find the sequence of random numbers generated from a seed of 0.192743568.

Solution:

0.14, 0.76, 0.15, 0.35, 0.62, 0.54, 0.62, 0.91, 0.48, 0.24, ...

CHI-SQUARE EVALUATION

This program calculates the value of the χ^2 statistic for the goodness of fit test by the equation

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

where O_i = observed frequency

E_i = expected frequency.

The χ^2 statistic measures the closeness of the agreement between the observed frequencies and expected frequencies.

Notes:

1. In order to apply this test to a set of given data, it may be necessary to combine some classes to make sure that each expected frequency is not too small (say, not less than 5).

2. If the expected frequencies E_i are all equal to some value E, then E should be computed beforehand as

$$E = \frac{\Sigma O_i}{n}$$

and then input at each step as the expected frequency E_i.

LINE	CODE	KEY ENTRY
00		
01	00	0
02	23 00	STO 0
03	23 01	STO 1
04	74	R/S
05	23 02	STO 2
06	41	–
07	15 02	g x^2
08	24 02	RCL 2
09	71	÷
10	23 51 01	STO + 1
11	24 00	RCL 0
12	01	1
13	51	+
14	23 00	STO 0
15	13 04	GTO 04
16	23 02	STO 2
17	41	–
18	15 02	g x^2
19	24 02	RCL 2
20	71	÷
21	23 41 01	STO – 1
22	24 00	RCL 0
23	01	1
24	41	–
25	23 00	STO 0
26	13 04	GTO 04
27		
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R_0 n
R_1 χ^2
R_2 E_i
R_3
R_4
R_5
R_6
R_7

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM	R/S		0.00
3	Perform for i = 1,..., n:						
	Input observed and expected						
	frequencies	O_i	↑				
		E_i	R/S				i
4	Delete erroneous data	O_k	↑				
		E_k	GTO	16	R/S		
5	Display χ^2		RCL	1			χ^2
6	For new case, go to step 2.						

Example:

O_i	8	50	47	56	5	14
E_i	9.6	46.75	51.85	54.4	8.25	9.15

Solution:

$\chi^2 = 4.84$

PAIRED t STATISTIC

Given a set of paired observations from two normal populations with means μ_1, μ_2 (unknown)

x_i	x_1	x_2	...	x_n
y_i	y_1	y_2	...	y_n

let

$$D_i = x_i - y_i$$

$$\overline{D} = \frac{1}{n} \sum_{i=1}^{n} D_i$$

$$s_D = \sqrt{\frac{\Sigma D_i^2 - \frac{1}{n}(\Sigma D_i)^2}{n-1}}$$

$$s_{\overline{D}} = \frac{s_D}{\sqrt{n}}$$

The test statistic

$$t = \frac{\overline{D}}{s_{\overline{D}}},$$

which has n − 1 degrees of freedom (df), can be used to test the null hypothesis

$$H_0: \mu_1 = \mu_2.$$

LINE	CODE	KEY ENTRY
00		
01	41	–
02	25	Σ+
03	13 00	GTO 00
04	14 22	f s
05	24 03	RCL 3
06	14 02	f $\sqrt{x}$
07	71	÷
08	14 21	f $\bar{x}$
09	21	x⇄y
10	71	÷
11	74	R/S
12	24 03	RCL 3
13	01	1
14	41	–
15	13 00	GTO 00
16		
17		
18		
19		
20		
21		
22		
23		
24		
25		
26		
27		
28		
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33		
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35		
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37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R_0	
R_1	
R_2	
R_3	n
R_4	Used
R_5	Used
R_6	ΣD_i
R_7	ΣD_i^2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Perform for i = 1,..., n:						
	Input one pair of observations	x_i	↑				
		y_i	R/S				i
4	Delete erroneous data	x_k	↑				
		y_k	–	f	Σ–		
5	Compute t and df		GTO	04	R/S		t
			R/S				df
6	For new case, go to step 2.						

Example:

x_i	14	17.5	17	17.5	15.4
y_i	17	20.7	21.6	20.9	17.2

Solution:

t = –7.16

df = 4.00

t STATISTIC FOR TWO MEANS

Suppose $\{x_1, x_2, ..., x_{n_1}\}$ and $\{y_1, y_2, ..., y_{n_2}\}$ are independent random samples from two normal populations having means μ_1, μ_2 (unknown) and the same unknown variance σ^2.

We want to test the null hypothesis

$$H_0: \mu_1 - \mu_2 = D$$

where D is a given number.

Define

$$\overline{x} = \frac{1}{n_1} \sum_{i=1}^{n_1} x_i$$

$$\overline{y} = \frac{1}{n_2} \sum_{i=1}^{n_2} y_i$$

$$t = \frac{\overline{x} - \overline{y} - D}{\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}} \sqrt{\dfrac{\Sigma x_i^2 - n_1 \overline{x}^2 + \Sigma y_i^2 - n_2 \overline{y}^2}{n_1 + n_2 - 2}}}$$

We can use this t statistic, which has the t distribution with $n_1 + n_2 - 2$ degrees of freedom, to test the null hypothesis H_0.

LINE	CODE	KEY ENTRY
00		
01	24 03	RCL 3
02	23 00	STO 0
03	24 06	RCL 6
04	23 01	STO 1
05	14 21	f $\bar{x}$
06	23 02	STO 2
07	34	CLX
08	23 03	STO 3
09	23 06	STO 6
10	23 07	STO 7
11	74	R/S
12	31	↑
13	14 21	f $\bar{x}$
14	51	+
15	24 02	RCL 2
16	21	x⇄y
17	41	–
18	24 00	RCL 0
19	15 22	g 1/x
20	24 03	RCL 3
21	15 22	g 1/x
22	51	+
23	14 02	f $\sqrt{x}$
24	71	÷

LINE	CODE	KEY ENTRY
25	24 01	RCL 1
26	24 02	RCL 2
27	15 02	g x^2
28	24 00	RCL 0
29	61	x
30	41	–
31	24 06	RCL 6
32	51	+
33	14 21	f $\bar{x}$
34	15 02	g x^2
35	24 03	RCL 3
36	61	x
37	41	–
38	24 00	RCL 0
39	24 03	RCL 3
40	51	+
41	02	2
42	41	–
43	71	÷
44	14 02	f $\sqrt{x}$
45	71	÷
46	13 00	GTO 00
47		
48		
49		

REGISTERS	
R_0	n_1
R_1	Σx^2
R_2	$\bar{x}$
R_3	n_2
R_4	Used
R_5	Used
R_6	Σy^2
R_7	Σy

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG			
3	Perform for i = 1,..., n_1 :						
	Input x-value	x_i	Σ+				i
4	Initialize for y		f	PRGM	R/S		0.00
5	Perform for i = 1,..., n_2 :						
	Input y-value	y_i	Σ+				i
6	Input D and compute t	D	R/S				t
7	To find the means of x- and y-						
	values		RCL	2			$\bar{x}$
			f	$\bar{x}$			$\bar{y}$
8	For a new case, go to step 2.						

Example:

x: 79, 84, 108, 114, 120, 103, 122, 120

y: 91, 103, 90, 113, 108, 87, 100, 80, 99, 54

$n_1 = 8$

$n_2 = 10$

D = 0 (i.e., $H_0 : \mu_1 = \mu_2$)

Solution:

t = 1.73

$\overline{x} = 106.25$

$\overline{y} = 92.50$

ONE SAMPLE TEST STATISTICS FOR THE MEAN

For a normal population (x_1, x_2 ..., x_n) with a known variance σ^2, a test of the null hypothesis

$$H_0: \text{mean } \mu = \mu_0$$

is based on the z statistic (which has a standard normal distribution)

$$z = \frac{\sqrt{n}\ (\bar{x} - \mu_0)}{\sigma}.$$

If the variance σ^2 is unknown, then

$$t = \frac{\sqrt{n}\ (\bar{x} - \mu_0)}{s}$$

is used instead. This t statistic has the t distribution with n − 1 degrees of freedom. $\bar{x}$ and s are the sample mean and standard deviation.

DISPLAY LINE	DISPLAY CODE	KEY ENTRY
00		
01	14 21	f $\bar{x}$
02	21	x⇄y
03	41	−
04	24 03	RCL 3
05	14 02	f $\sqrt{x}$
06	61	x
07	23 00	STO 0
08	34	CLX
09	74	R/S
10	24 00	RCL 0
11	14 22	f s
12	71	÷
13	74	R/S
14	24 00	RCL 0
15	21	x⇄y
16	71	÷
17	13 00	GTO 00
18		
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49		

REGISTERS
R_0 $\sqrt{n}\ (\bar{x} - \mu_0)$
R_1
R_2
R_3 n
R_4 Used
R_5 Used
R_6 Σx
R_7 Σx^2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG			
3	Perform for i = 1,..., n:						
	Input value	x_i	Σ+				i
4	Input μ_0	μ_0	f	PRGM	R/S		0.00
5	Compute t		GTO	10	R/S		t
	or						
5	Input σ and compute z	σ	GTO	14	R/S		z
6	For new case, go to step 2.						

Example:

Suppose $\mu_0 = 2$, for the following set of data

$$\{2.73, 0.45, 2.52, 1.19, 3.51, 2.75, 1.79, 1.83, 1, 0.87, 1.9, 1.62, 1.74, 1.92, 1.24, 2.68\}$$

Solution:

test statistic t = –.69

or z = –.57 if σ = 1.