Coupled Harmonic Oscillators — Theory

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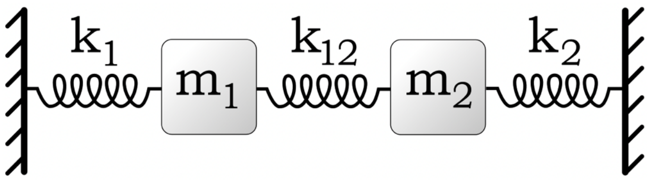
It will not become obvious for a while, but we are actually leaving oscillation and starting into waves. Our first step down this path is to do two oscillators. Our next step will be to do many oscillators (like 10 or 100).

General Case

Here is a nice diagram (thank you, Jim Belk and Wikimedia Commons) of two oscillators, coupled to each other:



Two harmonic oscillators, having two different masses, each coupled to a wall by two different springs, and then coupled to each other by one additional spring, is the most general coupled harmonic oscillator configuration.

Some Simplifications and Comparisons

There is no need to be so general on our first problem! Let us simplify! We can take $m_1 = m_2 = m$.

We can also take $k_1 = k_2 = k$.

We could even take $k_{12} = k$, but let's not do that, because it is interesting to see what happens when the oscillators are weakly coupled. In other words, when $k_{12} \ll k$.

Even though the two masses are now identical, we are going to need two positions and two velocities to describe the motion of the two masses.

So the code is going to be about as complicated as when you solved the Naval Battle notebook. In that notebook, there was only one mass (the cannonball), but it had two coordinates. Now there are two masses, but they each only have one coordinate, so once again we will have two coordinates and two velocities, but now for the completely different reason that there are two masses.

The Forces

The Left Spring Affects the First Mass

Let's let x_1 be the displacement (to the right) of the first mass (from its equilibrium position). How much is the left spring stretched? It is stretched by x_1.

It pulls in the negative direction if the first mass is moved in the positive direction (stretch), and it pushes in the positive direction if the first mass is moved in the negative direction (compressed). Both the compression the left-most spring on the first mass are neatly summarized by the single equation:

$$F_{\text{left spring on }1^{\text{st}}\text{ mass}} = -k x_1$$

It is super-important to pay attention to the signs, otherwise instead of oscillation we would get run-away exponential growth.

The Right Spring Affects the Second Mass

Let's do another easy one. The force of the right-most spring on the second mass is:

$$F_{\text{right spring on }2^{\text{nd}}\text{ mass}} = -k x_2$$

The Middle Spring Affects Both Masses

Now for the tricky forces that come from the coupling spring (the middle spring).

It is *stretched* if the 2nd mass is moved in the positive direction. But it is *compressed* if the first mass moves in the positive direction. So how much is the *net* amount of stretch?? $x_2 - x_1$!

When it is stretched it pulls in the positive direction on the first mass. So the force of the middle spring on the first mass is (note the plus sign):

$$F_{\text{middle spring on }1^{\text{st}}\text{ mass}} = +k_{12}\,(x_2 - x_1)$$

Also, when it is stretched it pulls in the negative direction on the second mass. So the force of the middle spring on the second mass is:

$$F_{\text{middle spring on }2^{\text{nd}}\text{ mass}} = -k_{12}\,(x_2 - x_1)$$

The Accelerations

As always we have $F = ma$, but now we have to apply Newton's Second Law to each of the masses.

So now there are two acceleration formulas:

$a_1 = F_1 / m$

$a_2 = F_2 / m$

We are almost there. We just have to put the total force on the first mass into the equation for a_1,

$a_1 = F_1 / m = \left(F_{\text{left spring on 1}^{\text{st}}\text{ mass}} + F_{\text{middle spring on 1}^{\text{st}}\text{ mass}}\right) / m = [-k x_1 + k_{12}(x_2 - x_1)] / m$

and the total force on the second mass into the equation for a_2:

$a_2 = F_2 / m = \left(F_{\text{right spring on 2}^{\text{nd}}\text{ mass}} + F_{\text{middle spring on 2}^{\text{nd}}\text{ mass}}\right) / m = [-k x_2 - k_{12}(x_2 - x_1)] / m$

Convenient Definitions

Sure, it is getting a little messy, but look on the bright side. The forces do not depend on time. And the forces do not depend on velocity! They only depend on positions. So coding up Runge-Kutta for both of these masses is not going to be as bad as it could be.

Also, we can make things look a tad simpler by defining:

$\omega_0{}^2 \equiv k / m$

$\omega_{12}{}^2 \equiv k_{12} / m$

Then we can write things with less parentheses and brackets:

$a_1 = -\omega_0{}^2 x_1 + \omega_{12}{}^2 (x_2 - x_1)$

$a_2 = -\omega_0{}^2 x_2 - \omega_{12}{}^2 (x_2 - x_1)$

Comments About Weak Coupling and No Coupling

If the coupling spring were not present, we would have two independent harmonic oscillators. Can we see that? Of course! If the coupling spring were not present, that is the same as setting $k_{12} = 0$ which is the same thing as setting $\omega_{12}{}^2 = 0$, and then we would just have:

$a_1 = -\omega_0{}^2 x_1$

$a_2 = -\omega_0{}^2 x_2$

Since we have simplified our lives a bit by not having any damping (no γ's), we know the exact solution in the case of no coupling:

$x_1(t) = A_1 \cos\omega_0 t + B_1 \sin\omega_0 t$

$x_2(t) = A_2 \cos\omega_0 t + B_2 \sin\omega_0 t$

The four constants can be anything. A_1 and A_2 are the initial displacements from equilibrium. $B_1\, \omega_0$ and $B_2\, \omega_0$ are the initial velocities.

Perhaps the weak coupling case bears some resemblance to the no-coupling case. Let's code it all up and find out what emerges.

Epilog — A Taste of What is Coming

The notebook you are about to do only has two masses and three springs. Can you see what we might have to do if we had seven masses and eight springs, and while we are generalizing, let's make all the springs identical to reduce the work a bit:

$a_1 = -\omega_0{}^2 x_1 + \omega_0{}^2 (x_2 - x_1)$

$a_2 = -\omega_0{}^2 (x_2 - x_1) + \omega_0{}^2 (x_3 - x_2)$

$a_3 = -\omega_0{}^2 (x_3 - x_2) + \omega_0{}^2 (x_4 - x_3)$

$a_4 = -\omega_0{}^2 (x_4 - x_3) + \omega_0{}^2 (x_5 - x_4)$

$a_5 = -\omega_0{}^2 (x_5 - x_4) + \omega_0{}^2 (x_6 - x_5)$

$a_6 = -\omega_0{}^2 (x_6 - x_5) + \omega_0{}^2 (x_7 - x_6)$

$a_7 = -\omega_0{}^2 (x_7 - x_6) - \omega_0{}^2 x_7$

The only ones that break the pattern a little are the end masses that are connected to the walls. All the interior masses follow a completely repetitive pattern. The two exceptions to the pattern are that the left-most mass doesn't have a left neighbor, and the right-most mass doesn't have a right neighbor.