Damped Oscillator to Damped Pendulum — Theory

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Damped Oscillator

We have contemplated an oscillator with the force law given by Hooke's Law for springs and a simple model of damping, and in the last notebook, we even included an external sinusoidal driving force. Without the external sinusoidal driving force, the force law is:

$$F(x, v) = -k x - b v$$

Divide through by the mass to find the acceleration. It is:

$$a(x, v) = -\frac{k}{m} x - \frac{b}{m} v$$

Let ${\omega_0}^2 = \frac{k}{m}$ and $\gamma = \frac{b}{2 m}$. The acceleration is then:

$$a(x, v) = -{\omega_0}^2 x - 2 \gamma v$$

Exact Theoretical Solution of the Damped Oscillator

Up until this point, we are still in the realm of problems that have exact solutions, and the most general solution of this equation I will just tell you is exactly known and it is:

$$x(t) = A \cos \sqrt{{\omega_0}^2 - \gamma^2}\, t \cdot e^{-\gamma t} + B \sin \sqrt{{\omega_0}^2 - \gamma^2}\, t \cdot e^{-\gamma t}$$

The oddball frequency that shows up in the cosine and sine functions is the somewhat slowed down oscillation rate due to the damping. It is necessary that γ (the decay rate due to damping) is less than ω_0, otherwise the oscillator is overdamped and the solutions are different in the overdamped case.

Because it can be hard to keep the symbols straight, it is good to summarize a significant feature of these solutions, which are the three frequencies:

ω_0	the natural frequency of the oscillator in the absence of damping
γ	the frequency that controls the rate of decay of the exponential
$\sqrt{{\omega_0}^2 - \gamma^2}$	the frequency of the oscillator including damping (which slows oscillation down)

These are all frequencies (e.g., they all have units of inverse time when you put the units back in).

Damped Pendulum

Angle, Angular Velocity, and Angular Acceleration

The angle of the pendulum we denote as θ. Typically for a pendulum, one chooses $\theta = 0$ to be straight down, and a rotation in the counter-clockwise direction to be the positive θ direction. Back when we first defined velocity, we had

$$v_{i \to i+1,\text{avg}} \equiv \frac{\text{change in position}}{\text{change in time}} = \frac{x(t_{i+1}) - x(t_i)}{\Delta t}$$

So we do the same thing again. We will call angular velocity ω. The equation analogous to the preceding one is:

$$\omega_{i \to i+1,\text{avg}} \equiv \frac{\text{change in angle}}{\text{change in time}} = \frac{\theta(t_{i+1}) - \theta(t_i)}{\Delta t}$$

Back when we introduced acceleration, we had

$$a_{i \to i+1,\text{avg}} \equiv \frac{\text{change in velocity}}{\text{change in time}} = \frac{v(t_{i+1}) - v(t_i)}{\Delta t}$$

We will call angular acceleration α. (Now you know why I had to change alpha to lambda in Generalized Runge-Kutta!) The equation analogous to the preceding one is:

$$\alpha_{i \to i+1,\text{avg}} \equiv \frac{\text{change in angular velocity}}{\text{change in time}} = \frac{\omega(t_{i+1}) - \omega(t_i)}{\Delta t}$$

Everything else that we did as we were deriving Second-Order Runge-Kutta — and I mean everything — is now completely applicable! You just make the replacements:

$x \to \theta$
$v \to \omega$
$a \to \alpha$

We just need the pendulum force law to put into Runge-Kutta, and I will give that to you next.

Pendulum Angular Acceleration per Newton’s Laws

The pendulum force law is:

$$F(\theta, \omega) = -mg\sin\theta - bl\omega$$

Technically, this is the “tangential force.” Divide through by the mass, m, and you get the tangential acceleration. Also divide through by the length, l, and then you get the angular acceleration:

$\alpha(\theta, \omega) = -\frac{g}{l}\sin\theta - \frac{b}{m}\omega$

Define $\omega_0^2 = \frac{g}{l}$, $\gamma = \frac{b}{2m}$, and you have:

$\alpha(\theta, \omega) = -\omega_0^2 \sin\theta - 2\gamma\omega$

Compare this with the acceleration for the damped harmonic oscillator:

$a(x, v) = -\omega_0^2 x - 2\gamma v$

You'll see that there are only two things different: (1) in angular problems, we use θ, ω, and α to describe the motion, instead of x, v, and a, and (2) there is a pesky $\sin\theta$ instead of a plain old θ. If the pesky $\sin\theta$ were instead θ, it would just be:

$\alpha(\theta, \omega) = -\omega_0^2 \theta - 2\gamma\omega$

Approximate Solution of the Damped Pendulum

We know the most general theoretical solution for the equation that has θ instead of $\sin\theta$! We just make the replacement $x \to \theta$, and get:

$\theta(t) = A\cos\sqrt{\omega_0^2 - \gamma^2}\, t \cdot e^{-\gamma t} + B\sin\sqrt{\omega_0^2 - \gamma^2}\, t \cdot e^{-\gamma t}$

Sadly, this is only an approximate solution because $\sin\theta$ is very close to θ only when θ is small. So we can describe the pendulum very well using the theoretical solution, provided the oscillation is not large.

ω_0	natural frequency of the pendulum in the absence of damping, *when oscillation is small*
γ	frequency that controls the rate of decay of the exponential, *when oscillation is small*
$\sqrt{\omega_0^2 - \gamma^2}$	frequency of the pendulum including damping, *when oscillation is small*

Exact Solution of the Damped Pendulum?

It is the fact that there is a $\sin\theta$ in the problem that makes the damped pendulum our first problem that we ***must*** do on the computer if we want to do large oscillations. There is only an approximate analytical solution, and it gets worse and worse as the angle of oscillation gets larger and larger.

I have seen claims that the damped pendulum is solved, but looking for references, like this paper by Kim Johannessen (2014), "An analytical solution to the equation of motion for the damped nonlinear pendulum," I see that the abstract begins with "an analytical ***approximation***." [The bold italics are mine, of course.] This is a great problem to study with numerical analysis. And that is what we are going to do next in our damped pendulum and forced pendulum notebooks!

Summary of the Damped Pendulum

With ${\omega_0}^2 = \frac{g}{l}$, $\gamma = \frac{b}{2m}$, the pendulum angular acceleration is:

$$\alpha(\theta, \omega) = -{\omega_0}^2 \sin\theta - 2\gamma\omega$$

General Second-Order Runge-Kutta for Angular Problems

Instead of re-deriving everything we did to get general second-order Runge-Kutta, we can mindlessly make the substitutions:

$x \to \theta$
$v \to \omega$
$a \to \alpha$

The result is the general second-order Runge-Kutta procedure for angular problems:

$$t^* = t + \lambda \Delta t$$

$$\theta^* = \theta(t_i) + \omega(t_i) \cdot \lambda \Delta t$$

$$\omega^* = \omega(t_i) + \alpha(t_i, \theta(t_i), \omega(t_i)) \cdot \lambda \Delta t$$

$$t_{i+1} = t_i + \Delta t$$

$$\omega(t_{i+1}) = \omega(t_i) + \left(\left(1 - \frac{1}{2\lambda}\right) \alpha(t_i, \theta(t_i), \omega(t_i)) + \frac{1}{2\lambda} \alpha(t^*, \theta^*, \omega^*)\right) \cdot \Delta t$$

$$\theta(t_{i+1}) = \theta(t_i) + (\omega(t_i) + \omega(t_{i+1})) \frac{\Delta t}{2}$$

Notice that because we are now using α for angular acceleration, I had to change what we used to call α in t^*, x^*, and v^* to another letter, and I chose λ.