Double Pendulum — Theory

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My apologies for muddying the waters by previously calling this the "Compound Pendulum." That name is reserved for something else that frankly is not a very interesting variation on the simple pendulum.

Simple Pendulum — Angular Acceleration — Recap

The pendulum force law (with damping) was:

$$F(\theta, \omega) = -mg \sin\theta - bl\omega$$

Technically, this is the "tangential force." Divide through by the mass, m, and you get the tangential acceleration. Also divide through by the length, l, and then you get the angular acceleration:

$$\alpha(\theta, \omega) = -\frac{g}{l} \sin\theta - \frac{b}{m}\, \omega$$

Define $\omega_0{}^2 = \frac{g}{l}$, and $\gamma = \frac{b}{2\,m}$ and you have:

$$\alpha(\theta, \omega) = -\omega_0{}^2 \sin\theta - 2\,\gamma\omega$$

I have often been choosing $\omega_0 = 2\,\pi$ which makes the natural period for small oscillations 1 second (assuming we are imagining our unit of time to be the second).

Double Pendulum — Angular Accelerations

At the website, https://www.jhallard.com/blog/double_pendulum.html, the equations below are presented. Seniors in physics typically derive these equation and derive them again in their first year of grad school, and I am not going to derive them for you. Frankly, I haven't had time (yet) to review the derivation, and I may update this document with a little more explanation of the equations if I get time.

Anyway, here are what Newton's Laws give you for the double pendulum:

$$\omega_1' = \frac{-g(2m_1+m_2)\sin(\theta_1)-m_2 g \sin(\theta_1-2\theta_2)-2\sin(\theta_1-\theta_2)m_2(L_2\omega_2^2+L_1\omega_1^2\cos(\theta_1-\theta_2))}{L_1(2m_1+m_2(1-\cos(2(\theta_1-\theta_2))))}$$

$$\omega_2' = \frac{2\sin(\theta_1-\theta_2)(L_1\omega_1^2(m_1+m_2)+\cos(\theta_1)g(m_1+m_2)+L_2m_2\omega_2^2\cos(\theta_1-\theta_2))}{L_2(2m_1+m_2(1-cos(2(\theta_1-\theta_2))))}$$

In these equations $\omega_1{}'$ is what we call α_1 and $\omega_2{}'$ is what we call α_2.

Dividing numerator and denominator of the equation for ω_1' through by $m_1 L_1$, we get:

$$\alpha_1 = \frac{-\frac{g}{L_1}\left(2+\frac{m_2}{m_1}\right)\sin\theta_1 - \frac{m_2}{m_1}\frac{g}{L_1}\sin(\theta_1-2\,\theta_2) - 2\sin(\theta_1-\theta_2)\frac{m_2}{m_1}\left(\frac{L_2}{L_1}\omega_2{}^2+\omega_1{}^2\cos(\theta_1-\theta_2)\right)}{2+\frac{m_2}{m_1}(1-\cos 2(\theta_1-\theta_2))}$$

Also dividing numerator and denominator of the equation for ω_2' through by $m_1 L_1$, we get:

$$\alpha_2 = \frac{2\sin(\theta_1-\theta_2)\left(\omega_1{}^2\left(1+\frac{m_2}{m_1}\right)+\cos\theta_1\frac{g}{L_1}\left(1+\frac{m_2}{m_1}\right)+\frac{L_2}{L_1}\frac{m_2}{m_1}\cos(\theta_1-\theta_2)\right)}{\frac{L_2}{L_1}\left(2+\frac{m_2}{m_1}(1-\cos 2\,(\theta_1-\theta_2))\right)}$$

I am going to choose some simplifying values. See the diagram at the end of this document. I am going to choose $\frac{m_2}{m_1}=\frac{1}{3}$, $\frac{L_2}{L_1}=\frac{1}{4}$, and $\frac{g}{L_1}=4\,\pi^2$ so that the natural period for small oscillations of the longer of the two pendulums is 1 second and the shorter pendulum has period $\frac{1}{2}$ second. With these values, the equations simplify to:

$$\alpha_1 = \frac{-28\,\pi^2\sin\theta_1 - 4\,\pi^2\sin(\theta_1-2\,\theta_2) - 2\sin(\theta_1-\theta_2)\left(\frac{1}{4}\omega_2{}^2+\omega_1{}^2\cos(\theta_1-\theta_2)\right)}{7-\cos 2(\theta_1-\theta_2))}$$

$$\alpha_2 = \frac{8\sin(\theta_1-\theta_2)\left(4\,\omega_1{}^2+16\,\pi^2\cos\theta_1+\frac{1}{4}\cos(\theta_1-\theta_2)\right)}{7-\cos 2\,(\theta_1-\theta_2)}$$

Second-Order Runge-Kutta — Formulas for Two Particles — Recap

For two oscillators, we had two positions, two velocities, and two acceleration formulas:

$$t_{i+1} = t_i + \Delta t$$

$$x_1{}^* = x_1(t_i) + v_1(t_i)\cdot\frac{\Delta t}{2}$$

$$x_2{}^* = x_2(t_i) + v_2(t_i)\cdot\frac{\Delta t}{2}$$

$$v_1{}^* = v_1(t_i) + a_1(x_1(t_i),\, x_2(t_i),\, v_1(t_i),\, v_2(t_i))\cdot\frac{\Delta t}{2}$$

$$v_2{}^* = v_2(t_i) + a_2(x_1(t_i),\, x_2(t_i),\, v_1(t_i),\, v_2(t_i))\cdot\frac{\Delta t}{2}$$

$$v_1(t_{i+1}) = v_1(t_i) + a_1(x_1{}^*,\, x_2{}^*,\, v_1{}^*,\, v_2{}^*)\cdot\Delta t$$

$$v_2(t_{i+1}) = v_2(t_i) + a_2(x_1{}^*,\, x_2{}^*,\, v_1{}^*,\, v_2{}^*)\cdot\Delta t$$

$$x_1(t_{i+1}) = x_1(t_i) + (v_1(t_i) + v_1(t_{i+1}))\,\frac{\Delta t}{2}$$

$$x_2(t_{i+1}) = x_2(t_i) + (v_2(t_i) + v_2(t_{i+1}))\,\frac{\Delta t}{2}$$

Second-Order Runge-Kutta — Double Pendulum

So now our job is just to translate this for the double pendulum, using mindless substitutions:

$x_1 \to \theta_1 \quad x_2 \to \theta_2$
$v_1 \to \omega_1 \quad v_2 \to \omega_2$
$a_1 \to \alpha_1 \quad a_2 \to \alpha_2$

$t_{i+1} = t_i + \Delta t$

$\theta_1{}^* = \theta_1(t_i) + \omega_1(t_i) \cdot \frac{\Delta t}{2}$

$\theta_2{}^* = \theta_2(t_i) + \omega_2(t_i) \cdot \frac{\Delta t}{2}$

$\omega_1{}^* = \omega_1(t_i) + \alpha_1(\theta_1(t_i), \theta_2(t_i), \omega_1(t_i), \omega_2(t_i)) \cdot \frac{\Delta t}{2}$

$\omega_2{}^* = \omega_2(t_i) + \alpha_2(\theta_1(t_i), \theta_2(t_i), \omega_1(t_i), \omega_2(t_i)) \cdot \frac{\Delta t}{2}$

That takes care of the starred values. Now for the final values:

$\omega_1(t_{i+1}) = \omega_1(t_i) + \alpha_1(\theta_1{}^*, \theta_2{}^*, \omega_1{}^*, \omega_2{}^*) \cdot \Delta t$

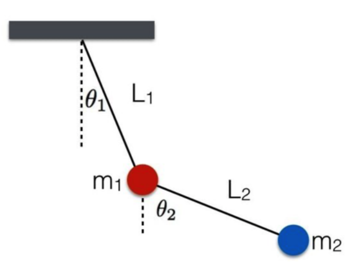
$\omega_2(t_{i+1}) = \omega_2(t_i) + \alpha_2(\theta_1{}^*, \theta_2{}^*, \omega_1{}^*, \omega_2{}^*) \cdot \Delta t$

$\theta_1(t_{i+1}) = \theta_1(t_i) + (\omega_1(t_i) + \omega_1(t_{i+1})) \frac{\Delta t}{2}$

$\theta_2(t_{i+1}) = \theta_2(t_i) + (\omega_2(t_i) + \omega_2(t_{i+1})) \frac{\Delta t}{2}$

A Visual Summary

In case you have lost the picture of what all these variable refer to, here is a diagram that has them labeled:



We have chosen the blue mass to be one-third of the red mass, and L_2 to be one-quarter of L_1.