Drumheads — Theory

April 1, 2025

Let's attempt to consolidate our understanding or at least record in one place the drumhead theory that we have been using.

This is a work in progress that I will have delayed presenting until April 1.

The Second Derivative — Recap

At the end of the theory that we developed for the second derivative, the wave equation looked like this:

$$\frac{\partial^2 \theta}{\partial t^2} = {v_0}^2 \frac{\partial^2 \theta}{\partial x^2}$$

The constant v_0 had the units of velocity.

Now we are leaving torsion waves, and moving to drumheads. So instead of an angle being dependent on t and x our first change is that we will have a z displacement that depends on t and x. So the first change is simply a renaming of the dependent variable:

$$\frac{\partial^2 z}{\partial t^2} = {v_0}^2 \frac{\partial^2 z}{\partial x^2}$$

Introducing the Second Dimension — Rectangular Case

The waves on a drumhead can propagate in any direction (the x direction, the y direction, and any other angle in the $x - y$ plane).

I can explain where the following comes from more, and indeed it isn't very hard to see, but first I will just write down the two-dimensional wave equation without explanation:

$$\frac{\partial^2 z}{\partial t^2} = {v_0}^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right)$$

The dependent variable is z. The independent space variables are x and y. As always, time t is an independent variable. Our job will be to get Mathematica to predict the z's as a function of x, y, and t.

Discretizing (Gridding) the Rectangular Case

What is our expression that we are

$$a_j(t_i) = v_0{}^2\ \frac{z_{j+1}(t_i) - 2\, z_j(t_i) + z_{j-1}(t_i)}{w_x{}^2}$$

I decided to rename Δx to w_x where w is short for "width." This is the x spacing between z_{j+1} and z_j. When we add in the second dimension, we have

$$a_j(t_i) = v_0{}^2\ \frac{z_{j+1}(t_i) - 2\, z_j(t_i) + z_{j-1}(t_i)}{w_x{}^2}$$

These equations tell how to get the θ values at a later time from the θ values at the current and previous times. If you don't see this, let me re-arrange so that you can see that this is really a way of stepping forward in time:

$$\theta_j(t_{i+1}) = 2\, \theta_j(t_i) - \theta_j(t_{i-1}) + \omega_0{}^2 (\Delta t)^2\ \frac{\theta_{j+1}(t_i) - 2\, \theta_j(t_i) + \theta_{j-1}(t_i)}{(\Delta x)^2}$$

ns write down "wave equations" this is the notation they use.

Polar Coordinates — Circular Case

If we describe the drumhead in terms of r and ϕ instead of x and y, but the displacement is still in the z direction, the equation becomes

$$\frac{\partial^2 z}{\partial t^2} = v^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\ \frac{\partial z}{\partial r} + \frac{1}{r^2}\ \frac{\partial^2 z}{\partial \phi^2}\right)$$

I can explain where the first and third term come from pretty easily. The middle term that only has one derivative is trickier. You have to imagine a cone-shaped z displacement and realize that even if there is zero second derivative in the r direction (z is linear in r), and only constancy in the ϕ direction (no dependence of z on ϕ), that a cone shaped drumhead will want to flatten itself out, and the linear derivative is what accounts for that.

Discretizing the Rectangular Case

What is our expression that we are

$$\frac{\theta_j(t_{i+1}) - 2\, \theta_j(t_i) + \theta_j(t_{i-1})}{(\Delta t)^2} = \omega_0{}^2\ \frac{\theta_{j+1}(t_i) - 2\, \theta_j(t_i) + \theta_{j-1}(t_i)}{(\Delta x)^2}$$

These equations tell how to get the θ values at a later time from the θ values at the current and previous times. If you don't see this, let me re-arrange so that you can see that this is really a way of stepping forward in time:

$$\theta_j(t_{i+1}) = 2\, \theta_j(t_i) - \theta_j(t_{i-1}) + \omega_0{}^2 (\Delta t)^2\ \frac{\theta_{j+1}(t_i) - 2\, \theta_j(t_i) + \theta_{j-1}(t_i)}{(\Delta x)^2}$$

ns write down "wave equations" this is the notation they use.

Discretizing (Gridding) the Circular Case