Torsion Waves — Theory

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We are headed into the last of three notebooks on torsion pendula, and by the time we we are done with the last one it will become obvious that we are finally seeing waves.

Torsion Pendulum — Theory

The theory we developed for the torsion pendulum and promptly used in our eleventh notebook was $\tau = -\kappa_L\,\theta - \kappa_R\,\theta$ and $\tau = I\alpha$.

The formula that plays the role of $F = ma$ but involves torque is read, “torque, τ, equals the moment of inertia, I, times angular acceleration, α.” I want to stress that this law involving torques is not an independent law. In a mechanics course, you would derive this law from Newton’s Second Law.

We put those two equations together to get:

$$\alpha = (-\kappa_L\,\theta - \kappa_R\,\theta)/I$$

For this problem, we defined

$$\omega_0{}^2 \equiv \frac{\kappa_L + \kappa_R}{I}$$

and our formula was then elegantly written as

$$\alpha = -\omega_0{}^2\,\theta$$

Coupled Torsion Pendula — Theory

The theory we developed for coupled torsion pendula and promptly used in our twelfth notebook was:

$$\alpha_1 = -\omega_0{}^2\theta_1 + \omega_{12}{}^2(\theta_2 - \theta_1)$$

$$\alpha_2 = -\omega_0{}^2\theta_2 - \omega_{12}{}^2(\theta_2 - \theta_1)$$

where this time we defined:

$$\omega_0{}^2 \equiv \frac{\kappa}{I} \text{ and } \omega_{12}{}^2 \equiv \frac{\kappa_{12}}{I}$$

Five Rods — Theory — Fixed Ends

Now we just need to graduate from one or two rods connected by stainless steel wires to seventy-two rods. Of course, seventy-two is not a magic number, and we will want to generalize to *n* rods, where *n* is anything from from 1 to 1000, or whatever you like.

Let’s write down the equations for five rods with six wires connecting them to each other and to the walls. That should be enough to see where we need to go:

$$\alpha_1 = -\omega_0{}^2(\theta_1 - 0) + \omega_0{}^2(\theta_2 - \theta_1)$$

$$\alpha_2 = -\omega_0{}^2(\theta_2 - \theta_1) + \omega_0{}^2(\theta_3 - \theta_2)$$

$$\alpha_3 = -\omega_0{}^2(\theta_3 - \theta_2) + \omega_0{}^2(\theta_4 - \theta_3)$$

$$\alpha_4 = -\omega_0{}^2(\theta_4 - \theta_3) + \omega_0{}^2(\theta_5 - \theta_4)$$

$$\alpha_5 = -\omega_0{}^2(\theta_5 - \theta_4) + \omega_0{}^2(0 - \theta_5)$$

where $\omega_0{}^2 \equiv \frac{\kappa}{I}$.

The left-most and right-most rods have equations that don’t quite fit the pattern due to being on the ends. This is assuming there are ends that can’t move. Their lack of movement is captured in the 0’s that I have suggestively placed into the amount the two end stainless steel wires are twisted.

Five Rods — Theory — Free Ends

If there are no wires at the ends, we call that “free” instead of “fixed,” and the equations are:

$$\alpha_1 = 0 + \omega_0{}^2(\theta_2 - \theta_1)$$

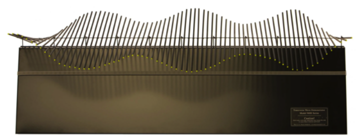
$$\alpha_2 = -\omega_0{}^2(\theta_2 - \theta_1) + \omega_0{}^2(\theta_3 - \theta_2)$$

$$\alpha_3 = -\omega_0{}^2(\theta_3 - \theta_2) + \omega_0{}^2(\theta_4 - \theta_3)$$

$$\alpha_4 = -\omega_0{}^2(\theta_4 - \theta_3) + \omega_0{}^2(\theta_5 - \theta_4)$$

$$\alpha_5 = -\omega_0{}^2(\theta_5 - \theta_4) + 0$$

If you look at the apparatus that is usually set up for physics demonstration, it is usually set up with free ends, and so maybe that is the more interesting case for us to code up first.



However, there are little clips that come with the apparatus so that you can fix the ends.

It is nice to understand both cases, because the way waves reflect off of free ends is quite different than the way they reflect off of fixed ends.

Summary

This angular acceleration formula

$$\alpha_j = -\omega_0{}^2(\theta_j - \theta_{j-1}) + \omega_0{}^2(\theta_{j+1} - \theta_j)$$

is valid except for the ends, and we have to handle those separately.

Fixed Ends

In the fixed-end case, the left-most rod’s angular acceleration is

$$\alpha_1 = -\omega_0{}^2(\theta_1 - 0) + \omega_0{}^2(\theta_2 - \theta_1)$$

and the right-most rod’s angular acceleration is

$$\alpha_n = -\omega_0{}^2(\theta_n - \theta_{n-1}) + \omega_0{}^2(0 - \theta_n)$$

Free Ends

In the free-end case, the left-most rod’s angular acceleration is

$$\alpha_1 = 0 + \omega_0{}^2(\theta_2 - \theta_1)$$

and the right-most rod’s angular acceleration is

$$\alpha_n = -\omega_0{}^2(\theta_n - \theta_{n-1}) + 0$$