General Second-Order Runge-Kutta — Damped Oscillation

Done in class, January 31, 2025

This is the fourth notebook for you to finish in-class.

Damped Oscillator

Problem Description

```
springConstant = 20; dampingConstant = 1;
force[x_, v_] := -springConstant x - dampingConstant v
mass = 5;
a[x_, v_] := force[x, v] / mass;
tInitial = 0;
tFinal = 10 Pi;
steps = 4800;
deltaT = (tFinal - tInitial) / steps;
```

Initial Conditions

Let's stretch this spring to $x_{\text{initial}} = 25$ and let it go with no initial velocity, so $v_{\text{initial}} = 0.0$.

In[ ]:=
```
xInitial = 25.0;
vInitial = 0.0;
initialConditions = {tInitial, xInitial, vInitial};
```

General Second-Order Runge-Kutta — Theory — Summary

This is a more general version of Second-Order Runge-Kutta, which has a parameter α, typically chosen as $\alpha = \frac{1}{2}$ or $\alpha = 1$:

$$t^* = t + \alpha\Delta t$$

$$x^* = x(t_i) + v(t_i) \cdot \alpha\Delta t$$

$$v^* = v(t_i) + a(t_i, x(t_i), v(t_i)) \cdot \alpha\Delta t$$

$$t_{i+1} = t_i + \Delta t$$

$$v(t_{i+1}) = v(t_i) + \left(\left(1 - \frac{1}{2\alpha}\right) a(t_i, x(t_i), v(t_i)) + \frac{1}{2\alpha} a(t^*, x^*, v^*)\right) \cdot \Delta t$$

$$x(t_{i+1}) = x(t_i) + (v(t_i) + v(t_{i+1})) \frac{\Delta t}{2}$$

General Second-Order Runge-Kutta — Implementation

```
alpha = 1;
rungeKutta2[cc_] := (
  (* Extract time, position, and velocity from the list. *)
  {newTime, newPosition, newVelocity}
 )
```

Displaying Damped Oscillation

Nest the procedure and then transpose the results to produce position and velocity plots:

```
In[ ]:= rk2Results = NestList[rungeKutta2, initialConditions, steps];
rk2ResultsTransposed = Transpose[rk2Results];
positionPlot = ListPlot[Transpose[rk2ResultsTransposed[[{1, 2}]]]]
```

```
In[ ]:= positions = rk2ResultsTransposed[[2]];
```

```
In[ ]:= Animate[NumberLinePlot[positions[[step]], PlotRange → {-25, 25}], {step, 0, steps, 1}]
```

Conclusion / Commentary

Our oscillator now has the force law $F(x) = -20\,x - v$. 20 was the spring constant and 1 was the damping constant. Nowhere did we put sines or cosines or decaying exponential functions into the problem! Damped oscillation (for which the position is the product of a cosine and a decaying exponential) emerged from the force law.

For damped oscillation, it is common to define the following ratios, both of which have the dimension of frequency when you put units back in:

```
In[24]:= omega0 = Sqrt[springConstant / mass]
gamma = dampingConstant / (2 mass)
```

Out[24]=

2

Out[25]=

$\frac{1}{10}$

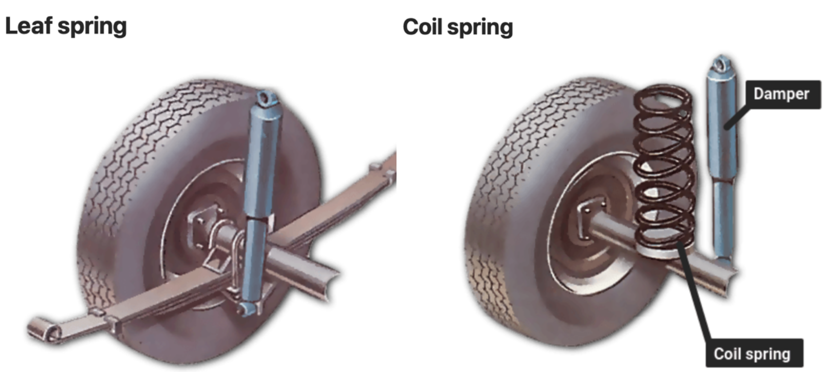
It turns out (and we can experiment with the parameters to actually see this) that if $\omega_0 > \gamma$, that the system is “underdamped.” The underdamped situation is what I chose for you when I chose 20 and 1 for the constants with a mass of 5. If $\omega_0 < \gamma$ the system is “overdamped.”

An overdamped system does not oscillate! No matter how much you stretch the spring and no matter how much initial velocity you give the mass, it always comes slowly to rest.

Epilog — Car Suspension

Car suspension is usually set up to be just a little underdamped, meaning that $\gamma < \omega_0$, but not a lot less. The left front suspension (right when viewed from our angle) of the vehicle in this video is a bit too underdamped: https://youtu.be/_7TPAp9fYL8. The left front shock absorber needs to be adjusted or replaced. It is unlikely that there is anything wrong with the spring, but nowadays, you may have to replace the whole “strut” which includes the spring if the shock absorber has worn out. One of you should show this page to Paul, and see if I have said anything terribly wrong from the perspective of a mechanic who actually knows how cars nowadays are typically built.

A simple mental model for a shock absorber in a car is just a piston that is closely fitted into a cylinder of oil. However, a modern shock absorber is actually more complicated, and has one-way valves to make compression and lengthening of the shock behave differently. In fact, the whole setup has gotten way more complicated over time, with the shock absorber originally being completely separated from a leaf spring, and then the next evolution was to have it working along-side a coil spring:



Finally, here is what I mentioned above, where the shock absorber and the spring are integrated into a strut: