Coupled Harmonic Oscillators

Completed and Analyzed in class, February 18, 2025

This is the eighth notebook for you to complete.

Coupled Oscillators — Formulas for the Accelerations

After all was said and done developing the theory, we were down to the following two simple equations for the accelerations,

$$a_1 = -\omega_0{}^2 x_1 + \omega_{12}{}^2 (x_2 - x_1)$$

$$a_2 = -\omega_0{}^2 x_2 - \omega_{12}{}^2 (x_2 - x_1)$$

where,

$$\omega_0{}^2 \equiv k / m$$

$$\omega_{12}{}^2 \equiv k_{12} / m$$

Coupled Oscillators — Implementing the Accelerations

In[60]:=
```
omega0 = 2 Pi; (* this value is chosen so
 that the period with no coupling is 1 second *)
omega12 = 0.2 omega0; (* this value is chosen so that the coupling is weak *)

a1[x1_, x2_] := bunny
a2[x1_, x2_] := rabbit
```

Initial Conditions

Let’s push the left mass to the left, and leave the right mass at its equilibrium. Let’s start both masses with no velocity:

In[64]:=
```
tInitial = 0.0;
tFinal = 60.0;
initialx1 = -0.5;
initialx2 = 0.0;
initialv1 = 0.0;
initialv2 = 0.0;
(* We have to decide how we are packing the time,
the two positions, and the two *)
(* velocities into the initial conditions
 list. How about this as one of two obvious *)
(* choices
  (and I really can't see that one choice is any better than the other): *)
initialConditions = {tInitial, initialx1, initialx2, initialv1, initialv2};
```

Second-Order Runge-Kutta — Formulas for Two Particles

As on Exam 1, just to cut down on the work, let's set $\lambda = \frac{1}{2}$, which simplifies the Second-Order Runge-Kutta formulas:

$$t_{i+1} = t_i + \Delta t$$

$$x_1{}^* = x_1(t_i) + v_1(t_i) \cdot \frac{\Delta t}{2}$$

$$x_2{}^* = x_2(t_i) + v_2(t_i) \cdot \frac{\Delta t}{2}$$

$$v_1(t_{i+1}) = v_1(t_i) + a_1(x_1{}^*, x_2{}^*) \cdot \Delta t$$

$$v_2(t_{i+1}) = v_2(t_i) + a_2(x_1{}^*, x_2{}^*) \cdot \Delta t$$

$$x_1(t_{i+1}) = x_1(t_i) + (v_1(t_i) + v_1(t_{i+1})) \frac{\Delta t}{2}$$

$$x_2(t_{i+1}) = x_2(t_i) + (v_2(t_i) + v_2(t_{i+1})) \frac{\Delta t}{2}$$

Second-Order Runge-Kutta — Implementing the Formulas

In[71]:=
```
steps = 12 000;
deltaT = (tFinal - tInitial) / steps;

rungeKutta2[cc_] := (
  curTime = cc[[1]];
  curx1 = squirrel;
  newTime = curTime + deltaT;
  (* Now I have left you six equations to implement *)
  chipmunk;
  chimpanzee;
  chihuahua;
  chipotle;
  macdonalds;
  wendys;
  {newTime, newx1, newx2, newv1, newv2}
 )
(* Test your implementation on the initial conditions *)
rungeKutta2[initialConditions]
(* I know I am notoriously wrong on these, but I actually check *)
(* pretty carefully. Sigh. Anyway, this is what I get: *)
(* {0.005,-0.499743,-9.8696 10^-6,0.102644,-0.00394784} *)
```

Using NestList[] to Repeatedly Apply rungeKutta2[]

In[75]:=
```
rk2Results = NestList[rungeKutta2, initialConditions, steps];
```

Transposing to Get Points We Can Put in ListLinePlot[]

In[76]:=
```
rk2ResultsTransposed = Transpose[rk2Results];
times = rk2ResultsTransposed[[1]];
x1s = bananas;
x2s = apples;
timesAndx1s = timesAndBananas;
timesAndx2s = timesAndApples;
```

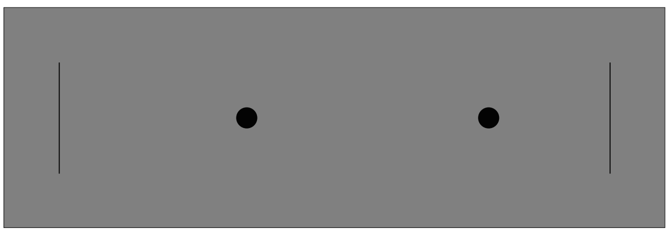
Plotting The Results

In[82]:=
```
ListLinePlot[{timesAndx1s, timesAndx2s}]
```

A Simple Graphic

In[83]:=
```
coupledOscillatorGraphic[x1_, x2_] := Graphics[{
   (* the first line makes a gray rectangle *)
   {EdgeForm[Thin], Gray, Polygon[{{0, -1}, {6, -1}, {6, 1}, {0, 1}}]},
   Line[{{0.5, -0.5}, {0.5, 0.5}}],
   Line[{{5.5, -0.5}, {5.5, 0.5}}],
   Style[Point[{x1 + 2, 0}], PointSize[0.03]],
   Style[Point[{x2 + 4, 0}], PointSize[0.03]]
  }]
coupledOscillatorGraphic[0.2, 0.4]
```

Out[84]=



Animating The Graphics

In[85]:=
```
Animate[coupledOscillatorGraphic[x1s[[i]], x2s[[i]]],
 {i, 1, steps, 1}, DefaultDuration → tFinal - tInitial]
```

Comparing With YouTube

YouTube could be a great reference, but it has filled up with such an amazing amount of garbage, that finding a good video of coupled harmonic oscillators was surprisingly hard. I finally found a whole page of demonstrations put together by Caltech: https://physicsdemos.caltech.edu/index_simple.html, and one of the demonstrations was of the situation that we have simulated:

https://youtu.be/Eoux0MsZqBY

Just watch the first little bit. Then, if we want to understand more of what is going on later in the video, we are going to have to put in a bit of damping and a driving force, but we have already done so much of the fundamental work in building up this notebook, that would not actually be very difficult.