Quantum Physics, Preparation for Tuesday, Mar. 12

In Moore, Finish Q9 and Start Q10

Study Moore through p. 154. You should have *deja vu* while reading g p. 154, because you did it as Problem 2(d) on the Feb. 20th exam. This would be a great time to re-do Problem 2 if you missed any part of it, or even if you didn't miss anything. What you did in Problem 2 is central to everything else we will do with wave functions.

For Problem Set 10

In the last problem set, you did Q9M.5 on p. 148. Eq. Q9.21 was:

$$\int_{-\infty}^{\infty} e^{-x^2/a^2}\, dx = a\sqrt{\pi}$$

Another utterly equivalent way of writing Q9.21 is:

$$\int_{-\infty}^{\infty} e^{-x^2/2b}\, dx = \sqrt{2\pi b}$$

and that is the way I usually try to memorize it because most closely corresponds to a neat trick for deriving the integral. If you guys ask me, I will show you the trick. Or you can just take it on faith.

Yet another way of writing the integral is:

$$\int_{-\infty}^{\infty} e^{-cx^2}\, dx = \sqrt{\pi/c}$$

1. In the above, c is a constant, and x is the dummy variable of integration. However, the formula is valid for any c, so you can think of it as a variable too.

(a) Thinking of c as a variable and taking $\frac{\partial}{\partial c}$ of both sides of the above equation, what new integral do you have the answer for?

(b) Here is an easy integral to do:

$$\int_{-\infty}^{\infty} x e^{-cx^2}\, dx$$

What is it?

2. Thanks to Problem 1, you now know the integrals for

$$\int_{-\infty}^{\infty} x^n e^{-c x^2} dx$$

with $n = 0, 1,$ and 2. You can of course put in $c = 1/a^2$ and then you know the integrals for

$$\int_{-\infty}^{\infty} x^n e^{-x^2/a^2} dx$$

with $n = 0, 1,$ and 2.

(a) Normalize the wave function $\psi(x) = N e^{-x^2/2 a^2}$. In other words, find N. Do not forget to begin by squaring $\psi(x)$. Normalization is always done with the probability, not the probability amplitude. I put in the 2 for your convenience. When you square, it goes away.

(b) Using the normalized wave function, calculate

$$\int_{-\infty}^{\infty} x \,|\, \psi(x) \,|^2 dx$$

This is the “expectation value” of x. It is, on average, what the electron’s x-coordinate is. The notation for the average x-coordinate is $\overline{x}$.

(c) Again using the normalized wave function, calculate

$$\int_{-\infty}^{\infty} x^2 \,|\, \psi(x) \,|^2 d\, x$$

This is the expectation value, or the average measurement of x^2. The notation for the average measurement of x^2 is $\overline{x^2}$.

NOTE: $\overline{x^2}$ and $\overline{x}^2$ are completely different, right?

(d) Whatever you got for (c), take the square root.

NOTE: You now know $\sqrt{\overline{x^2}}$. This slightly whacko combination is known to statisticians as the “standard deviation” of x, and statisticians generally denote it as σ, but we aren’t statisticians.

3. Let’s do some more integrals. This would be a good time to review changes of variables and integration by parts, if you don’t remember how those techniques are used to do integrals.

(a) Using the same N as you previously had, what is

$$\int_{-\infty}^{\infty} N^2 e^{-(x-x_0{}^2)/a^2}\, dx$$

HINT AND DOUBLE-CHECK: There is a simple change of variables that reduces this to an integral you already calculated. The result is super-simple! Perhaps it is obvious why.

(b) What is

$$\overline{x} = \int_{-\infty}^{\infty} x\, N^2\, e^{-(x-x_0)^2/a^2}\, dx$$

HINT: The same simple change of variables applies, but this time, the result is not quite so simple.

(c) What is:

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2\, N^2\, e^{-(x-x_0)^2/a^2}\, dx$$

4. The momentum operator is $\frac{\hbar}{i}\frac{\partial}{\partial x}$.

(a) Calculate

$$\int_{-\infty}^{\infty} N^2\, e^{-(x-x_0)^2/2\,a^2}\, \frac{\hbar}{i}\frac{\partial}{\partial x}\, e^{-(x-x_0)^2/2\,a^2}\, dx$$

This is denoted $\overline{p}$.

(b) The momentum operator squared is $-\hbar^2\frac{\partial^2}{\partial x^2}$. Calculate

$$-\hbar^2 \int_{-\infty}^{\infty} N^2\, e^{-(x-x_0)^2/2\,a^2}\, \frac{\partial^2}{\partial x^2}\, e^{-(x-x_0)^2/2\,a^2}\, d\,x = \hbar^2 \int_{-\infty}^{\infty} N^2 \left(\frac{\partial}{\partial x}\, e^{-(x-x_0)^2/2\,a^2}\right)\left(\frac{\partial}{\partial x}\, e^{-(x-x_0)^2/2\,a^2}\right) d\,x$$

This is denoted $\overline{p^2}$. I used integration by parts to get the second form of the integral, because it makes the algebra a tad easier.

5. In this problem you will derive a version of the Heisenberg Uncertainty Principle. You'll do the special case with $x_0 = 0$.

(a) Take your answer for 3(c) with $x_0 = 0$. In the special case that $\overline{x} = 0$, which is the case you are dealing with now, $\sqrt{\overline{x^2}}$ gets its own symbol. It is Δx. What is Δx?

NOTE: Now you know part of what goes into Eq. Q9.17.

(b) Take your answer to 4(b). In the special case that $\overline{p} = 0$, which is the case you are dealing with right now, $\sqrt{\overline{p^2}}$ also gets its own symbol. It is Δp. What is Δp?

(c) Well, you have Δx and Δp. What is $\Delta x \cdot \Delta p$?

NOTE: Compare with Eq. 9.17. Note that the Heisenberg Uncertainty Principle has an inequality in it. The function you used, a Gaussian, is the best that can be done. For any other function, $\Delta x \cdot \Delta p$ is larger.

FINAL NOTE: I will introduce a non-zero p_0 and allow x_0 to be non-zero as well, and derive the slightly more general version of the result that you just got in class on Tuesday, but I have to define and discuss Δx and Δp for the more general case first.