Modern Introductory Physics, Part II, Exam 2

Tuesday, Feb. 20, 2024 — Covering Churchill, Brown and Verhey, Sections 1-6, and Moore, Chapters Q6, Q7, and Q9.

I LEFT ENOUGH ROOM ON THIS EXAM TO ANSWER ON THE EXAM. ALL THE ALGEBRA STEPS ARE SHORT. IF YOU NEED SCRATCH PAPER, FEEL FREE, BUT PUT YOUR REASONING AND YOUR ANSWERS ON THE EXAM.

1. Complex Variables in Polar Form (4 pts)

(a) Graph the complex number $z = -1 + i$ using the usual conventions for graphing complex numbers you learned about in Section 5 of Churchill, Brown, and Verhey. Your graph will look a little like this figure from p. 11:

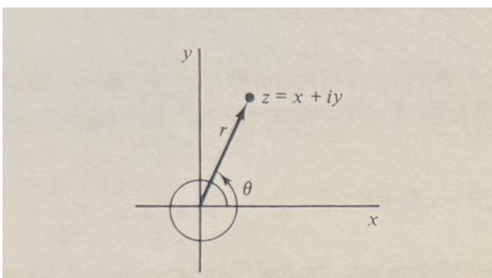

FIGURE 4

Be sure to label r, θ, x, and y on your diagram for $z = -1 + i$. There is enough room to the right of the figure above to make your diagram.

(b) What are r and θ? For θ, feel free to answer in either radians or degrees.

(c) So you now have $z = -1 + i$ in polar form, $z = re^{i\theta}$, use the polar form to calculate $w = z^6$. What simple expression you can give for w?

(d) Similarly, use the polar from to calculate $u = 1/z^8$. What simple expression you can give for u?

2. Working up to the Schrödinger Equation (4 pts)

On Problem Set 7, you simplified both sides of the equation $i\hbar \frac{\partial}{\partial t}\left(e^{-iEt/\hbar}\, e^{ipx/\hbar}\right) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\left(e^{-iEt/\hbar}\, e^{ipx/\hbar}\right)$. Consider the similar equation:

$$i\hbar \frac{\partial}{\partial t}\left(e^{-iEt/\hbar}\, \psi(x)\right) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\left(e^{-iEt/\hbar}\, \psi(x)\right).$$

(a) For this part, just simplify the left-hand side (LHS) of this equation as much as you can. Remember, $\frac{\partial}{\partial t}$ means to take a derivative with respect to time while pretending x is a constant.

(b) For this part, just simplify the right-hand side (RHS) of the equation as much as you can. Remember, $\frac{\partial^2}{\partial x^2}$ means to take two derivatives with respect to x while pretending t is a constant.

COMMENT: It might be annoying that you don't have any idea what the function $\psi(x)$ is yet, but we'll get to a specific case in Part (d).

(c) Set what you found for the LHS in (a) equal to what you found for the RHS in (b) and simplify as much as you can.

HINT: All remnants of the variable t should have gone away after simplifying. Check your derivatives and your algebra if t hasn't gone away.

(d) Now that we have what is called the "time-independent Schrödinger equation," we'll put a specific function in it. Consider this normalized wavefunction for a particle stuck in a box of length L,

$$\psi(x) = \sqrt{\frac{2}{L}}\, \sin \frac{n \pi x}{L}$$

In this formula, n is any positive integer, and we only care about the range $0 < x < L$. Put this $\psi(x)$ into the equation you found in (c). Simplify as much as you can. You should discover that the equation is satisfiable, but out of it comes an equation for E that only involves some constants, the positive integer n, and the variables m and L. What is your equation for E?

3. Stern-Gerlach — Classical (3 pts)

In the “Them’s the Rules” handout, I did a short summary of some stuff you would have learned if you had had a semester of electromagnetism before quantum mechanics. Specifically, I gave you this formula for the force on a dipole (there was a different formula for the torque):

$$F_i = \frac{\partial}{\partial x_i} \vec{\mu} \cdot \vec{B}$$

In the Stern-Gerlach experiment, we often choose the coordinates so that the magnetic field, $\vec{B}$, only has a z component. Then we just have:

$$F_z = \mu_z \frac{\partial B_z}{\partial z}$$

(a) Suppose B_z near the north pole of a magnet pointing in the $-z$ direction is b/z^5. What does the formula for F_z simplify to?

(b) Suppose $\mu_z = g \frac{-e}{2m} S_z$. With this substitution, what does the formula for F_z become?

(c) Plug in $S_z = +\frac{\hbar}{2}$ for a spin up electron. Now what do you find for F_z?

COMMENT: You can see why Moore draws his Stern-Gerlach diagrams with the magnet’s north pole pointing down and the beam passing below the magnet. He arranged it this way so that for positive S_z he gets a positive F_z.

4A. Stern-Gerlach — Basic Quantum (5 pts)

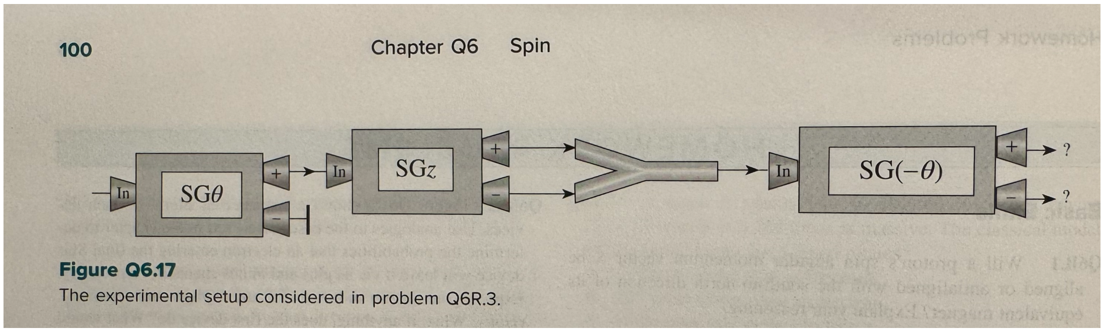

Figure Q6.17
The experimental setup considered in problem Q6R.3.

This is a diagram from near the end of Chapter Q6. For the last Stern Gerlach device, $(-\theta)$ doesn't mean what you might think!!

*By $(-\theta)$ with the parenthesis, Moore means that if the first Stern-Gerlach device is rotated **clockwise** around the beam direction by an amount θ, then the third Stern-Gerlach device is rotated around the beam by the same amount, but **counter-clockwise**. PLEASE BE SURE you understand the meaning of $(-\theta)$ with the parenthesis before proceeding.*

You are going to work in the z-basis and use the state vector notation to solve this problem.

In the diagram, you can see that the second Stern-Gerlach device is in the z-direction. But also, Moore has recombined its outputs using one of his Y-shaped combiners, ***so for this problem, the second Stern-Gerlach device has no effect.***

(a) In the state vector notation, $|+\theta\rangle$, the incoming state, is $\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$.

Simply plugging in $-\theta$ into the above state vector, what is the state vector notation for the state $|+(-\theta)\rangle$?

NOTE: You of course should simplify using the fact that cosine is an even function and sine is an odd function.

(b) Knowing that the second device has no effect and therefore can be ignored, what is the **amplitude** to get the $|+(-\theta)\rangle$ state out of the third device given that the first device produces the $|+\theta\rangle$ state? In other words, evaluate $\langle +(-\theta) \mid +\theta \rangle$.

(c) Further simplify your answer to (b) using $\cos^2\frac{\theta}{2} = \frac{1}{2}(1 + \cos\theta)$and $\sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta)$.

(d) Now square your answer to (c) get a **probability**, *P*.

(e) If *N* electrons come out of the first device how many electrons are expected to come out of the upper pipe of the third device? Also simplify your answer to for the special case that $\theta = 90°$.

4B. Stern-Gerlach — Surprising Quantum (5 pts)

Imagine the exact same setup as in Problem 4A, but with the |-z> output of the second Stern-Gerlach device blocked. Now the second device matters.

(a) In this new situation, we want to know the amplitude to get the |+(-*θ*)> state out of the third device given that this sequence of devices starts by preparing the |+*θ*> state, **but** now passes through a second device that only lets through the |+z> state? Simply write an expression of the form <blah|dee><da|bling>.

(b) Now use the state vector representations of <blah|, |dee>, <da|, and |bling> to get an expression for the **amplitude** you found in part (a).

(c) Use $\cos^2\frac{\theta}{2} = \frac{1}{2}(1 + \cos\theta)$ to simplify your answer to (b).

(d) Square your answer for the amplitude to get a **probability**, *P*.

(e) If *N* electrons come out of the first device how many electrons are expected to come out of the upper pipe of the third device? Also, simplify your answer to for the special case $\theta = 90°$.

COMMENT: Contrasting 4A(e) with 4B(e) is when quantum mechanics gets surprising.

5. Operators and States (4 pts)

Let’s evaluate $< -z \mid S_y \mid +z >$ using our usual tricks.

(a) Find | -z > in terms of | +y > and | -y > by solving

$$\mid +y > = \frac{1}{\sqrt{2}} (\mid +z > + i \mid -z >)$$

and

$$\mid -y > = \frac{1}{\sqrt{2}} (\mid +z > - i \mid -z >)$$

HINT: Subtracting the equations will immediately get rid of |+z>.

(b) Find | +z > in terms of | +y > and | -y >.

HINT: Adding the equations will immediately get rid of |-z>.

(c) Use what you got for | + z > in (b) to help you evaluate $S_y \mid +z >$. Of course, you’ll be using $S_y \mid +y > = \frac{\hbar}{2} \mid +y >$ and $S_y \mid -y > = -\frac{\hbar}{2} \mid -y >$.

(d) Now finish the job of evaluating $< -z \mid S_y \mid +z >$ by applying what you found above.

6. A Whacky **Extra Credit** Problem (2 pts)

If you manage to finish all the previous problems early, this will keep you entertained...

(a) What is $< +z \mid S_y \mid +z >$?

(b) What is $< +z \mid S_y^2 \mid +z >$?

NOTE: S_y^n means that you take the operator S_y and apply it n times. So $S_y^2 \mid +z >$ just means $S_y \, S_y \mid +z >$.

(c) What is $< +z \mid S_y^3 \mid +z >$?

(d) Are you starting to see a pattern? What is $< +z \mid S_y^n \mid +z >$?

Name ____________________

1. / 4

2. / 4

3. / 3

4A. / 5

4B. / 5

5. / 4

6. / 2 EC

GRAND TOTAL

/ 25 MAX