Modern Introductory Physics, Part II, Exam 3

Friday, Mar. 29, 2024 — Covering Moore Chapters 10-12.

YOU MAY NEED SOME SEPARATE PAPER TO WORK THE PROBLEMS. AT THE END, PLEASE STAPLE YOUR WORK TO THE EXAM AND TURN IT ALL IN.

1. Building a Table of Gaussian Integrals

Let's repeat and extend some of the things you did on Problem Set 10. First we need to rebuild our table of integrals.

(a) Start with $\int_{-\infty}^{\infty} e^{-c x^2}\, dx = \sqrt{\pi / c}$. Treating c as the variable, take $\frac{d}{dc}$ of each side of this equation and simplify to get a formula for $\int_{-\infty}^{\infty} x^2 e^{-c x^2}\, dx$.

(b) Take another derivative with respect to c and simplify to get a formula for $\int_{-\infty}^{\infty} x^4 e^{-c x^2}\, dx$.

(c) Summarize the results so you can use them in the next problem:

$\int_{-\infty}^{\infty} e^{-c x^2}\, dx = \sqrt{\pi / c}$

$\int_{-\infty}^{\infty} x^2 e^{-c x^2}\, dx =$

$\int_{-\infty}^{\infty} x^4 e^{-c x^2}\, dx =$

(d) Summarize the results again, but substituting $c = 1/b^2$,

$\int_{-\infty}^{\infty} e^{-x^2/b^2}\, dx =$

$\int_{-\infty}^{\infty} x^2 e^{-x^2/b^2}\, dx =$

$\int_{-\infty}^{\infty} x^4 e^{-x^2/b^2}\, dx =$

2. Normalization and Expectation Values

It can be important in determining whether quantum mechanics predictions are right (it is too much to be more specific here), to know things like $\overline{x^4} = \int_{-\infty}^{\infty} \psi^*(x)\, x^4\, \psi(x)\, d\,x = \int_{-\infty}^{\infty} x^4 \,|\, \psi(x) \,|^2\, d\,x$, or as an even more important example, $\overline{p^4} = \int_{-\infty}^{\infty} \psi^*(x) \left(\frac{\hbar}{i} \frac{d}{d\,x}\right)^4 \psi(x)\, d\,x$.

(a) For the ground state of the harmonic oscillator, $\psi(x) = c_0\, e^{-x^2/2\,b^2}$. Use your results from 1(d) to normalize $\psi(x)$, e.g., to determine c_0 by demanding $1 = \int_{-\infty}^{\infty} |\, \psi(x) \,|^2\, d\,x$.

(b) Use your results from 1(d) and 2(a) to determine

$\overline{x^2} = \int_{-\infty}^{\infty} x^2 \,|\, \psi(x) \,|^2\, d x$

(c) Use your results from 1(d) and 2(a) to determine

$\overline{x^4} = \int_{-\infty}^{\infty} x^4 \,|\, \psi(x) \,|^2\, d x$

(ASIDE: For those of you who have had or will have statistics, you need $\overline{x} = \int_{-\infty}^{\infty} x\, P(x)\, d\,x$, $\overline{x^2} = \int_{-\infty}^{\infty} x^2\, P(x)\, d\,x$, $\overline{x^3} = \int_{-\infty}^{\infty} x^3\, P(x)\, d\,x$, and $\overline{x^4} = \int_{-\infty}^{\infty} x^4\, P(x)\, d\,x$ to calculate the mean, variance, skew, and kurtosis of a distribution.)

3. Reflection at a Potential Jump

We are going to learn about reflection at jump in the potential.

(a) Imagine a de Broglie wave coming in from the left (e.g., from negative x) and hitting a barrier at $x = 0$.

$$\psi_L(x, t) = e^{i k_L x - i \omega t}$$

Using the time-dependent Schrödinger equation, $i\hbar \frac{\partial \psi_L(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_L(x,t)}{\partial x^2}$, what is the formula for k_L in terms of $\hbar$, ω, and m?

(b) We haven't normalized $\psi_L(x, t)$ and we can't. It represents an infinite stream of particles coming at the barrier from the left. There might be a reflected stream too. We can add this to $\psi_L(x, t)$ as follows:

$$\psi_L(x, t) = e^{i k_L x - i \omega t} + b\, e^{-i k_L x - i \omega t}$$

What does $\psi_L(x, t)$ simplify to at $x = 0$?

(c) Take $\frac{\partial \psi_L(x,t)}{\partial x}$.

(d) What does $\frac{\partial \psi_L(x,t)}{\partial x}$ simplify to at $x = 0$?

4. Transmission at a Potential Jump

This is actually a continuation of Problem 3. If there is a barrier, and it isn't too high, we expect there to be a transmitted wave going to the right.

(a) We can model that as

$\psi_R(x,\ t) = c\ e^{i\,k_R\,x - i\,\omega\,t}$

We'll have the barrier be V_0 high for $x > 0$, and we'll assume $\hbar\omega > V_0$.

(ASIDE: We assume $\hbar\omega > V_0$ because that is what "isn't too high" amounts to. We could alternatively assume $\hbar\omega < V_0$, in which case the region $x > 0$ would be classically forbidden. But that is a different problem.)

Using the time-dependent Schrödinger equation, $i\hbar\,\frac{\partial\psi_R(x,t)}{\partial t} = -\frac{\hbar^2}{2\,m}\,\frac{\partial^2\psi_R(x,t)}{\partial x^2} + V_0\,\psi_R(x,\ t)$, what is the formula for k_R in terms of $\hbar$, ω, m and V_0?

(b) What does $\psi_R(x,\ t)$ simplify to at $x = 0$?

(c) Take $\frac{\partial\psi_R(x,t)}{\partial x}$.

(d) What does $\frac{\partial\psi_R(x,t)}{\partial x}$ simplify to at $x = 0$?

(e) Using your answers for 3(b) and 4(b) what does continuity demand?

(f) Using your answers for 3(d) and 4(d) what does the no-kinks condition demand?

HINT/CROSS-CHECK: If you did the algebra and simplification right, for parts (e) and (f) you will have two equations for the two unknowns b and c. The equations will not involve x or t or ω. They just involve k_L and k_R (which in turn depend on all the other constants). If we wanted to continue and learn more about transmission and reflection, next we would solve these equations for b and c.

5. Harmonic Oscillators in Thermal Equilibrium

You got a taste of fermions in thermal equilibrium in the section on semiconductors. In this problem, we will work on the simpler situation of a lot of harmonic oscillators in thermal equilibrium.

To make life easier, instead of writing $\frac{1}{k_B T}$ a lot, we will define $\beta \equiv \frac{1}{k_B T}$.

The fundamental principle of statistical mechanics is that for a system in thermal equilibrium, the probability, P_n, that a system has energy E_n is proportional to $e^{-\beta E_n}$. We don't know the proportionality constant yet. I will just write it as $\frac{1}{Z}$, so $P_n = \frac{1}{Z} e^{-\beta E_n}$.

(a) For the harmonic oscillator $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$. What is $\frac{P_1}{P_0}$?

(b) What is $\frac{P_{n+1}}{P_n}$?

(c) To determine Z, it has to be that the sum of all the P_n adds up to 1. so $1 = \sum_{n=0}^{\infty} \frac{1}{Z} e^{-\beta E_n}$ or $Z = \sum_{n=0}^{\infty} e^{-\beta E_n}$. Put in $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$. Factor out the overall $e^{-\frac{1}{2}\beta\hbar\omega}$. You are close to knowing Z for the harmonic oscillator.

(d) Use $e^{-\beta\hbar\omega n} = \left(e^{-\beta\hbar\omega}\right)^n$ and $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ to finalize your answer for Z.

(e) Put your answer for Z into $P_n = \frac{1}{Z} e^{-\beta E_n}$ and simplify. When you are done simplifying, you will know P_n in terms of the constants β, $\hbar$, ω, n, and of course e.

Name ____________________

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2. / 5
3. / 5
4. / 5
5. / 5

GRAND TOTAL

/ 25