Fermion Gas

In Section 11.5, Moore states the Pauli Exclusion Principle. An immediate consequence of the principle is that it costs more and more energy to stuff additional fermions into a box, assuming the first ones occupied the lowest available energy states.

Of course, it costs even more energy if they are charged, because like charges repel, but we are ignoring the Coulomb repulsion. We just want to know the total energy cost to stuff N electrons into a box of length L on each side. To warm ourselves up to the 3-dimensional case, we'll do the 1-dimensional case first.

After doing our calculations, we will apply them to white dwarf stars. These are very dense stars containing only ordinary matter. We'll treat the white dwarf star as a box and simply estimate the energy required to have all the electrons in this box.

Ignoring the Coulomb repulsion is actually completely reasonable in the case of a white dwarf, because the electrons are intermingled with a roughly equal number of protons, and so the Coulomb repulsion is roughly balanced out by the Coulomb attraction.

The Pauli Exclusion Principle

The principle that determines the properties of a fermion gas is that "no two identical fermions can have exactly the same quantum state."

If we are only dealing with electrons, as Pauli was in 1925, the principle is more simply stated as "no two electrons can have exactly the same quantum state."

Defining Terms

We have been using "fermion" and "electron" interchangeably, but an electron is just one example of a fermion. Other examples of fermions are neutrons, protons, muons, and neutrinos. Any particle with spin 1/2 (or spin 3/2, 5/2, etc.) is a fermion.

Contrast this with the photon, which has spin 1. Particles with spin 0, spin 1 (or spin 2, 3, etc.) are called "bosons." As we were told in the lasing presentation, bosons "like" to be in the same state, where "like" means that there is an enhanced probability of them ending up in the same state. Another example of a boson is the Higgs particle.

Filling the 1-D Fermion Gas

The $n = 1$ state can hold 2 electrons. The $n = 2$ state can hold two more. So, if the electron states are filled up to $n = N$, we have $2N$ electrons, and the total energy cost of stuffing the $2N$ electrons into the box of length L is

$$E_{\text{TOTAL}} = 2\left(1 + 4 + 9 + \ldots + N^2\right)\frac{h^2}{8\,mL^2} = 2\,\frac{N\,(N+1)\,(2\,N+1)}{6}\,\frac{h^2}{8\,mL^2}$$

We have used the formula for the sum of squares.

The Large-N Approximation for Filling the 1-D Fermion Gas

If N is very large, it is reasonable to simplify $\frac{N\,(N+1)\,(2\,N+1)}{6}$ to just $\frac{N^3}{3}$ because if N is very large, then the N^2 and N^1 terms are negligible in comparison to the N^3 term. So our approximate answer is:

$$E_{\text{TOTAL}} = 2\,\frac{N^3}{3}\,\frac{h^2}{8\,mL^2}$$

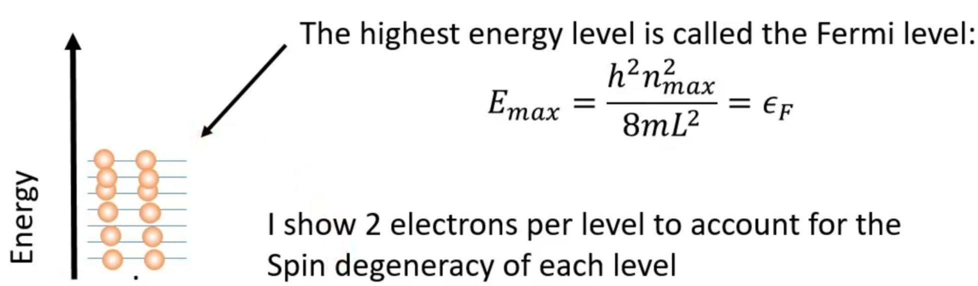
People notice that the energy to put the last electron in was $\frac{h^2\,N^2}{8\,mL^2}$ and they define that combination to be ϵ_F and call it “the Fermi energy.” So,

$$N = \left(\frac{8\,mL^2}{h^2}\,\epsilon_F\right)^{1/2}$$

and E_{TOTAL} can be rewritten as:

$$E_{\text{TOTAL}} = \frac{2}{3}\left(\frac{8\,mL^2}{h^2}\,\epsilon_F\right)^{3/2}\frac{h^2}{8\,mL^2} = \frac{2}{3}\left(\frac{8\,mL^2}{h^2}\right)^{1/2}\epsilon_F{}^{3/2}$$

In the diagram below, $n_{\text{max}} = 2\,N$, but in the 3-D case, which we’ll do next, n_{max} and N do not have such a simple relationship.



Filling the 3-D Fermion Gas

The 3-D case is distinctly harder than the 1-D case, because it isn’t obvious what energy levels get filled in what order. Consider the first level. It has $n_x = 1$, $n_y = 1$, and $n_z = 1$, and energy

$$\frac{3\,h^2}{8\,mL^2}$$

Where did the 3 come from!? It is $1^2 + 1^2 + 1^2$.

Of course, this level can hold two electrons. There are three second levels:

$n_x = 2, n_y = 1, n_z = 1$
$n_x = 1, n_y = 2, n_z = 1$
$n_x = 1, n_y = 1, n_z = 2$

They have energy

$$\frac{6 h^2}{8 m L^2}$$

Where did the 6 come from? It is $2^2 + 1^2 + 1^2$.

It isn't obvious what fills up next. Is it the 3 1 1 state and its friends, or is the the 2 2 1 state and its friends? Well, you have to compare $3^2 + 1^2 + 1^2 = 11$ with $2^2 + 2^2 + 1^2 = 9$. Ah-ha, 2 2 1 is next, and its friends 2 1 2, and 1 2 2. How about 2 2 2? Well, $2^2 + 2^2 + 2^2 = 12$ so 3 1 1 is next, and only after we have done 3 1 1, 1 3 1, and 1 1 3 will we get to 2 2 2.

This business of comparing sums of squares is going to turn into an awful mess. We need to make some kind of approximation that is only going to be valid when N is large, just like we did to get a tidy answer in the 1-D case.

The Large-N Approximation for Filling the 3-D Fermion Gas

In the preceding sub-section, we saw that the filling order is already not obvious in the 3-D case as soon as you get to 2 1 1, 3 1 1, or 2 2 2.

Here is an extremely creative approximation!

We notice that $n_x^2 + n_y^2 + n_z^2$ looks suspiciously similar to $x^2 + y^2 + z^2$ which we would call a radius-squared, or just r^2. Of course, it isn't the same at all, but this causes us to think of $n_x^2 + n_y^2 + n_z^2$ as a radius-squared, and what is happening is that the energy levels are filling up by increasing radius.

Furthermore! There is exactly 1 combination of n_x, n_y, and n_z for each little cube of volume 1 in the n_x-n_y-n_z space. So if we imagine a spherical shell going from r to $r + \Delta r$ then it contains $4 \pi r^2 \Delta r$ points on average. (Because the spherical shell's area is $4 \pi r^2$ and its thickness is Δr, and we just argued that there is on average one n_x-n_y-n_z combination in each unit volume of space.) The only correction to this is that we have to divide by 8 because we are only doing the octant with positive n_x, n_y, and n_z.

If we stuff the box full of fermions until there are $2N$ of them, they must fill up the n_x-n_y-n_z space out to a radius r and using the approximation just argued, we must have:

$$2N = 2\cdot\frac{1}{8}\int_0^r 4\pi s^2\, ds = \frac{1}{4}\cdot 4\pi\ \frac{r^3}{3}$$

Now we know how far out the n_x-n_y-n_z space is filled for a given N. As in the 1-D case, the energy to put the last electron in is

$$\epsilon_F = \frac{h^2 r^2}{8mL^2}$$

except that now instead of $n_{\max}$ it is r, the maximum radius, that shows up. So,

$$r = \sqrt{\frac{8mL^2\,\epsilon_F}{h^2}}$$

We can stick that into the relationship between N and r and we get

$$2N = \frac{1}{4}\cdot 4\pi\ \frac{1}{3}\left(\frac{8mL^2\,\epsilon_F}{h^2}\right)^{3/2}$$

The Energy of the 3-D Fermion Gas

$$E_{\text{TOTAL}} = 2\cdot\frac{1}{8}\int_0^r 4\pi s^2\ \frac{h^2 s^2}{8mL^2}\ ds = \frac{1}{4}\cdot 4\pi\ \frac{h^2}{8mL^2}\ \frac{r^5}{5}$$

That was straightforward, but we want to rewrite this using our relationship between r and ϵ_F. It is

$$E_{\text{TOTAL}} = \frac{1}{4}\cdot 4\pi\ \frac{1}{5}\ \frac{h^2}{8mL^2}\ \left(\frac{8mL^2\,\epsilon_F}{h^2}\right)^{5/2}$$

A popular way, because so much cancels(!), to write this is to calculate the average energy per fermion:

$$\frac{E_{\text{TOTAL}}}{2N} = \frac{\frac{1}{4}\cdot 4\pi\ \frac{1}{5}\ \frac{h^2}{8mL^2}\left(\frac{8mL^2\,\epsilon_F}{h^2}\right)^{5/2}}{\frac{1}{4}\cdot 4\pi\ \frac{1}{3}\left(\frac{8mL^2\,\epsilon_F}{h^2}\right)^{3/2}} = \frac{3}{5}\ \epsilon_F$$