Harmonic Oscillator Raising and Lowering Operators

We did some important things with the harmonic oscillator solutions. We normalized the first two of them. We built an oscillating wave packet out of the first two of them.

We have not shown that they are solutions, and we haven't gotten a taste of where the whacky recursion relations on p. 160 of Moore came from.

The Time-Independent Schrödinger Equation

The time-independent Schrödinger equation for the harmonic oscillator
is:

$$E_n\,\psi_n(x) = -\frac{\hbar^2}{2m}\,\frac{d^2\,\psi_n(x)}{d\,x^2} + \frac{1}{2}\,k_S\,x^2\,\psi_n(x)$$

Recall that this came from the full time-dependent Schrödinger Equation,

$$i\,\hbar\,\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\,\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x,\,t)\,\psi(x,\,t)$$

The only two things we did to get the time-independent equation from this equation were to put in the harmonic oscillator potential, which is time-independent, and is just

$$V(x) = \frac{1}{2}\,k_S\,x^2$$

and we put in the ansatz

$$\psi(x,\,t) = e^{-i\,E_n\,t/\hbar}\,\psi_n(x)$$

Scaling Energy by $\hbar\,\omega$

One thing we can do to make the equation slightly tidier is to define ω by $k_S = m\,\omega^2$. Then the time-independent equation is

$$E_n\,\psi_n(x) = -\frac{\hbar^2}{2m}\,\frac{d^2\,\psi_n(x)}{d\,x^2} + \frac{1}{2}\,m\,\omega^2\,x^2\,\psi_n(x)$$

We can measure the energy E_n in units of $\hbar\omega$. Then we have

$$E_n\ \hbar\omega\ \psi_n(x) = -\frac{\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} + \frac{1}{2} m\,\omega^2 x^2\,\psi_n(x)$$

or

$$E_n\ \psi_n(x) = \frac{1}{2}\left[-\frac{\hbar}{m\omega}\frac{d^2\psi_n(x)}{dx^2} + \frac{m\,\omega}{\hbar} x^2\,\psi_n(x)\right]$$

Scaling Length by $\hbar\,\omega$

Isn’t it interesting that the constants multiplying x^2, which is $\frac{m\,\omega}{\hbar}$, is exactly the reciprocal of the constant multiplying $\frac{d^2}{dx^2}$. This motivates us to notice that $\sqrt{\frac{\hbar}{m\,\omega}}$ has the dimension of length! If we measure x in units of $\sqrt{\frac{\hbar}{m\,\omega}}$, then we have

$$E_n\ \psi_n(x) = \frac{1}{2}\left[-\frac{d^2\psi_n(x)}{dx^2} + x^2\,\psi_n(x)\right]$$

The Solutions Moore Gave Us

On p. 160, Moore told us

$$E_n = \hbar\,\omega\left(n + \frac{1}{2}\right)$$

For n even, $\psi_n(x) = \left[c_0 + c_2 (b\,x)^2 + \ldots\right] e^{-(b x)^2/2}$

For n odd, $\psi_n(x) = \left[c_1\,b\,x + c_3 (b\,x)^3 + \ldots\right] e^{-(b x)^2/2}$

where there is some whacky recursion relation for the c_{k+2} in terms of c_k and $b = \sqrt{\frac{m\,\omega}{\hbar}}$.

The Rescaled Solutions

Now that we have scaled energy and length as we did in the preceding two subsections, Moore’s solutions simplify to

$$E_n = n + \frac{1}{2}$$

For n even, $\psi_n(x) = \left[c_0 + c_2\,x^2 + \ldots\right] e^{-x^2/2}$

For n odd, $\psi_n(x) = \left[c_1\,x + c_3\,x^3 + \ldots\right] e^{-x^2/2}$

We have no idea where the whacky recursion relations come from. We did calculate, using normalization, that $c_0 = \frac{1}{\pi^{1/4}}$, so

$$\psi_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2}$$

If you wished to restore the length that was scaled out, you would have

$$\psi_0(x) = \frac{1}{\pi^{1/4}} b^{1/2} e^{-(b\,x)^2/2}$$

but to keep our lives simple, we are going to keep working in terms of rescaled solutions.

Checking the Ground State

Show that

$$\psi_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2}$$

solves

$$E_0\ \psi_0(x) = \frac{1}{2}\left[-\frac{d^2\,\psi_0(x)}{d\,x^2} + x^2\ \psi_0(x)\right]$$

The factor of $\frac{1}{\pi^{1/4}}$ obviously doesn't affect whether or not the equation is solved, so leave it off to make your life easy. When you are done, what is E_0?

The Great Trick

I don't know who came up with this, but one of the best differential-equation solving tricks I have ever seen is this one, which works kind of like a proof by induction.

Assume that $\psi_n(x)$ exists and solves:

$$E_n\ \psi_n(x) = \frac{1}{2}\left[-\frac{d^2\,\psi_n(x)}{d\,x^2} + x^2\ \psi_n(x)\right]$$

with $E_n = n + \frac{1}{2}$

Show that:

$$\phi_{n+1}(x) \equiv \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}\psi_n}{\mathrm{dx}} + x\ \psi_n(x)\right)$$

solves

$$E_{n+1}\ \phi_{n+1}(x) = \frac{1}{2}\left[-\frac{d^2\,\phi_{n+1}(x)}{d\,x^2} + x^2\,\phi_{n+1}(x)\right]$$

with of course

$$E_{n+1} = n + \frac{3}{2}$$

When you are checking that it works, you can leave off the $\frac{1}{\sqrt{2}}$ in $\phi_{n+1}(x)$ to make your life easy.

The Meaning of the Great Trick

It must, from what you just showed, that given $\psi_n(x)$, you have discovered how to make $\psi_{n+1}(x)$!! The only thing you haven't figured out is how to normalize $\phi_{n+1}(x)$. I'll just tell you that the relationship is

$$\psi_{n+1}(x) = \frac{1}{\sqrt{n+1}}\,\phi_{n+1}(x) = \frac{1}{\sqrt{n+1}}\,\frac{1}{\sqrt{2}}\left(-\frac{d\psi_n(x)}{dx} + x\,\psi_n(x)\right)$$

Usually people put the $\sqrt{n+1}$ on the other side and write this as

$$\sqrt{n+1}\ \psi_{n+1}(x) = \frac{1}{\sqrt{2}}\left(-\frac{d\psi_n(x)}{dx} + x\,\psi_n(x)\right)$$

Because this operator tells you how to make $\psi_{n+1}(x)$ from $\psi_n(x)$ it is called the raising operator and given the symbol $A^\dagger$,

$$A^\dagger \equiv \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right)$$

so a very fancy way of writing what the raising operator does is

$$A^\dagger\,\psi_n(x) = \sqrt{n+1}\ \psi_{n+1}(x)$$

The First Excited State

Apply the trick with $n = 0$. In other words,
use

$$\psi_1(x) = \frac{1}{\sqrt{0+1}}\,\phi_{0+1}(x) = \frac{1}{\sqrt{0+1}}\,\frac{1}{\sqrt{2}}\left(-\frac{d\psi_0(x)}{dx} + x\,\psi_0(x)\right) = \frac{1}{\sqrt{2}}\left(-\frac{d\psi_0(x)}{dx} + x\,\psi_0(x)\right)$$

with $\psi_0(x) = \frac{1}{\pi^{1/4}}\,e^{-x^2/2}$ to discover the first excited state.

Companion to the Great Trick

It is similarly shown (doing the same things you did above) that there is a companion trick:

$$\chi_n(x) \equiv \frac{1}{\sqrt{2}}\left(\frac{d\psi_{n+1}(x)}{dx} + x\,\psi_{n+1}(x)\right)$$

or after putting in the normalization:

$$\psi_n(x) = \frac{1}{\sqrt{n+1}}\,\chi_n(x) = \frac{1}{\sqrt{n+1}}\,\frac{1}{\sqrt{2}}\left(\frac{d\psi_{n+1}(x)}{dx} + x\,\psi_{n+1}(x)\right)$$

Usually people put the $\sqrt{n+1}$ on the other side and write this as

$$\sqrt{n+1}\;\psi_n(x) = \frac{1}{\sqrt{2}}\left(\frac{d\psi_{n+1}(x)}{dx} + x\,\psi_{n+1}(x)\right)$$

Because this operator tells you how to make $\psi_n(x)$ from $\psi_{n+1}(x)$ it is called the lowering operator and it is given the symbol A,

$$A \equiv \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right)$$

and a fancy way of writing what this combination does is

$$A\,\psi_{n+1}(x) = \sqrt{n+1}\;\psi_n(x)$$

Usually people reduce n by 1 on both sides of this equation, and write what the lowering operator does as

$$A\,\psi_n(x) = \sqrt{n}\;\psi_{n-1}(x)$$

The State Below the Ground State

Apply A to $\psi_0(x) = \frac{1}{\pi^{1/4}}\,e^{-x^2/2}$ to discover the state below the ground state. Once you have an answer, argue that this had to be the answer.

One Last Trick

Using what you know, what is $A^\dagger A\psi_n(x)$? So from that, what is a snazzy way of writing $E_n\,\psi_n(x)$ using $A^\dagger A$?