Magnetic Resonance I — Setting Up the Problem

Magnetic resonance is the process of getting a spin to flip by applying an oscillating magnetic field to it.

An important piece of technology combining magnetic resonance and nuclear physics is nuclear magnetic resonance imaging (various acronyms for this are NMR, NMRI, and MRI). The spin that is flipped is the proton which is the simple nucleus of the hydrogen atom. Hydrogen is abundant in water and fat molecules, so a magnetic resonance image is a map of the density of water and fat which varies from tissue to tissue.

To understand the basic, brilliant idea underlying nuclear magnetic resonance imaging, you need to understand how you flip a spin by applying an oscillating magnetic field, and that is going to take us right back to spin-1/2 systems. Protons and electrons are both spin-1/2 particles, and either can be flipped using magnetic resonance, but in medical MRI machines, the resonance is tuned to flip the proton spin.

Review of Magnetic Moment

Back in Chapter 6, we learned that if something is charged and it is spinning, then it is also a little magnet. We quantify the strength and direction of the little magnet, in a quantity we call the magnetic moment, which in symbols is $\vec{\mu}$ and the direction of $\vec{\mu}$ is parallel to the spin $\vec{S}$ if the particle is positively charged, and negative if it is negatively charged. The proportionality constant includes a flexible fudge factor g which account for the fact that dimensional analysis and vector analysis doesn't tell us the exact relation. For the proton, which is positively charged, we have

$$\vec{\mu} = g \frac{e}{2m} \vec{S}$$

Where $e = 1.6 \times 10^{-19}$ C is a (positive) constant, m is the mass of the proton, and g, the fudge factor, is 5.6 for the proton.

Torque on a Magnetic Moment in a Magnetic Field

The torque on a magnetic moment in a magnetic field aligns the moment with the field.

If the field is in the z-direction, the magnitude of the torque is

$$|\vec{\tau}| = |\vec{\mu}|\,|\vec{B}| \sin\theta$$

where θ is how much the magnetic moment is tipped away from the z-axis.

Potential Energy of a Magnetic Moment in a Magnetic Field

The potential energy that corresponds to the above torque is

$$V = -|\vec{\mu}|\,|\vec{B}|\cos\theta$$

The minus sign is because the potential energy is lowest when the magnetic moment is aligned with the field ($\theta = 0\,°$), and highest when the magnetic moment is anti-aligned with the field ($\theta = 180\,°$).

It will occur to you that the above is the same as the dot product, so we could write

$$V = -\vec{\mu}\cdot\vec{B}$$

and combining that with $\vec{\mu} = g\,\frac{e}{2\,m}\,\vec{S}$, we have

$$V = -g\,\frac{e}{2\,m}\,\vec{S}\cdot\vec{B}$$

This is the vector way of writing the equation for V, and we can use it no matter which way $\vec{B}$ points.

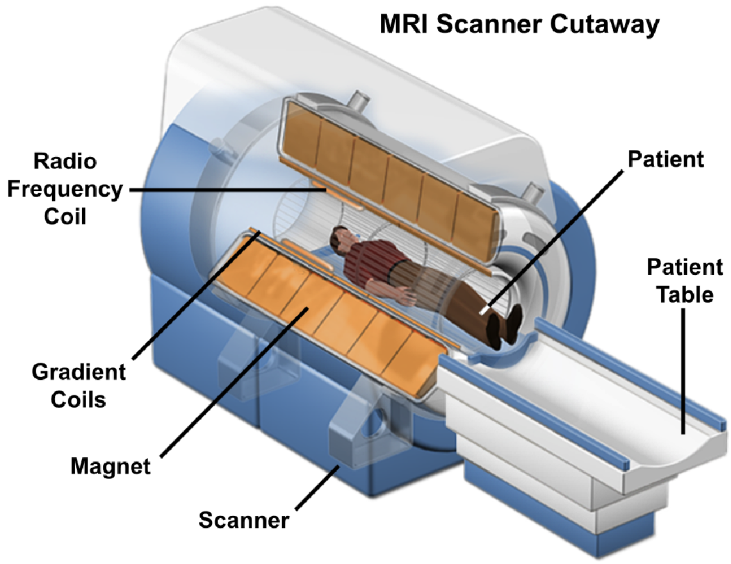
The Configuration of an MRI Machine

On the next page is an MRI scanner cutaway view taken from the National MagLab website. I don't particularly want to explain the "gradient coils." Those are used to slightly increase or decrease the strength of the overall magnetic field, and as these coils are adjusted, they are responsible for the banging noise that you hear when you are inside an MRI machine.

The "Radio Frequency Coil" is incredibly important but we won't get to that until the Section titled "The Resonance Idea" below.

For the moment just pay attention to the giant magnet, which is just labeled "Magnet." This is a toroidal magnet and the person is in the middle of it. The magnetic field goes from the feet to the head — or from the head to the feet. The direction (foot-to-head or head-to-foot) is not critical, just the axis, which is along the person's body.

We are going to make that the direction that the axial magnetic field points our +*z* axis.



Schrödinger’s Equation for a Magnetic Moment in a Magnetic Field

We aren’t going to worry about the proton moving around from one place to another. We are just going to study how the proton reorients itself in the magnetic field. In that case, $V = -g \frac{e}{2m} \vec{S} \cdot \vec{B}$ is everything we need to know about the particle’s energy and we can write down Schrodinger’s equation, it is

$$i\hbar \frac{d}{dt} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} = -g \frac{e}{2m} \vec{S} \cdot \vec{B} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix}$$

To make this equation a little more intelligible, let’s remember what it simplifies to when $\vec{B}$ has magnitude B_0 and points in the z-direction, which is how we are choosing to set up the axes in our MRI scanner problem. In that case Schrödinger’s equation is:

$$i\hbar \frac{d}{dt} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} = -g \frac{e}{2m} S_z B_0 \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} = -g \frac{e}{2m} B_0 \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix}$$

Let’s take the mess $g \frac{e}{2m} B_0$ and give it the name, ω_0. It is after all, a frequency (how can you quickly deduce that?). The $\hbar$ cancels and we just have

$$i \frac{d}{dt} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} = -\frac{\omega_0}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix}$$

This is the matrix way of writing two equations, and it is overly fancy because the matrix is diagonal. It is just two independent equations: $i \frac{d}{dt} \psi_+(t) = -\frac{\omega_0}{2} \psi_+(t)$ and $i \frac{d}{dt} \psi_-(t) = +\frac{\omega_0}{2} \psi_-(t)$.

The Resonance Idea

If the magnetic field pointed mostly in the z direction, *but also just a little in the x direction,* then we would have:

$$i\hbar \frac{d}{dt}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = -g\,\frac{e}{2m}\left(S_z B_0 + S_x B_1\right)\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix}$$

with $B_1 \ll B_0$. NOW THE EQUATION ISN'T SO SIMPLE because S_x has off-diagonal entries (see below). And the equation is about to get a little worse!

Here is the resonance idea: instead of just letting the magnetic field point a little in the x direction, let the amount that it points in the x direction oscillate with angular frequency ω! Then we have:

$$i\hbar \frac{d}{dt}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = -g\,\frac{e}{2m}\left(S_z B_0 + S_x B_1 \cos\omega t\right)\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix}$$

where B_0 and B_1 are constants and we still have $B_1 \ll B_0$, and the oscillation with angular frequency ω is accounted for in the $\cos\omega t$ factor.

Just like we took the mess $g\,\frac{e}{2m}\,B_0$ and gave it the name, ω_0, let's take the mess $g\,\frac{e}{2m}\,B_1$ and give it the name, ω_1. Then we have:

$$i\hbar \frac{d}{dt}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = \left(S_z\,\omega_0 + S_x\,\omega_1 \cos\omega t\right)\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix}$$

You probably remembered that $S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ when I used it in the previous section, but you may have forgotten that $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$. These matrices were consequences of the behavior of electrons in Stern-Gerlach apparatus. Go back to p. 108 in Chapter Q7 and Problem Q7D.3 on pp. 118-119 to refresh yourself. These choices aren't unique, it turns out. They are just the standard ones. Other choices give the same answers. We will just use these. Anyway, ...

Using the formulae for S_z and S_x, again the $\hbar$ cancels, and we have

$$i\frac{d}{dt}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = -\omega_0\,\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \omega_1 \cos\omega t\cdot\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}\omega_0 & \omega_1\cos\omega t\\ \omega_1\cos\omega t & -\omega_0\end{pmatrix}\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix}$$

This is the differential equation we have to solve. It turns out we can't solve it exactly, even with good tricks.