Magnetic Resonance II — Solving the Problem

In the previous handout, we set up the magnetic resonance problem, and on Problem Set 19, you did some interpretation of the $\omega_1 = 0$ behavior.

Now we are going to follow I.I. Rabi's method for solving the problem for non-zero ω_1. Although both the $\omega = \omega_0$ and ω near ω_0 cases are solvable using Rabi's method, we are just going to do $\omega = \omega_0$, because it is a bit simpler.

First we need to review so we can keep it all straight.

Keeping it All Straight

The last handout finished with this differential equation:

$$i \frac{d}{dt} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \omega_0 & \omega_1 \cos\omega t \\ \omega_1 \cos\omega t & -\omega_0 \end{pmatrix} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix}$$

Let's remind ourselves what is in it. $\psi_+(t)$ is the probability amplitude for finding the proton in the $+z$ state. If we solve the equation and find it, then $P_+(t) = |\psi_+(t)|^2$ will be the probability that the proton is in the $+z$ state (and similarly for $P_-(t) = 1 - P_+(t) = |\psi_-(t)|^2$).

The goal of magnetic resonance imaging is to get a lot of protons in your body first going to the $+z$ state and then to the $-z$ state and to then detect this by the radio signals that all these simultaneously oscillating protons collectively emit.

Part of keeping it all straight is to remember that there are three frequencies in the problem. The "zeroth" frequency came from the amount of magnetic field in the z direction, which is assumed to be large compared to the amount of magnetic field in the x direction. We had for the zeroth and first frequencies:

$$\omega_0 \equiv g \frac{e}{2m} B_0, \quad \omega_1 \equiv g \frac{e}{2m} B_1, \quad \omega_0 \gg \omega_1$$

The third frequency, ω, which has no subscript in our notation, is the angular frequency at which the magnetic field in the x direction oscillates.

Even if you can't keep it in your head how we got to this point, you need to remember that this is where we are, that we need to solve this equation, at least approximately, and thereby learn the probability amplitudes $\psi_+(t)$ and $\psi_-(t)$, and that once we do, we will know $P_+(t)$ and $P_-(t)$.

The $\omega_1 = 0$ Case

Problem 3 of Problem Set 19, was to solve the $\omega_1 = 0$ case. You found:

$$\begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} = \begin{pmatrix} \psi_+(0)\, e^{\frac{i}{2}\omega_0 t} \\ \psi_-(0)\, e^{-\frac{i}{2}\omega_0 t} \end{pmatrix}$$

The Ansatz

The next step is to return to the full case, $\omega_1 \neq 0$. Let's give the solutions we just found in the $\omega_1 = 0$ case some names. We can't call them $\psi_+(t)$ and $\psi_-(t)$ any more because now we are doing the full case, and those names should be reserved for the solutions. Let's call $a_+(t) = e^{\frac{i}{2}\omega_0 t}$ and $a_-(t) = e^{-\frac{i}{2}\omega_0 t}$.

Rabi's ansatz is to guess that $\psi_+(t) = a_+(t)\, c_+(t)$ and $\psi_-(t) = a_-(t)\, c_-(t)$ where $c_+(t)$ and $c_-(t)$ are expected to be slowly varying if $\omega_1 << \omega_0$. After all, if $\omega_1 = 0$, then $c_+(t)$ and $c_-(t)$ are just the constants $\psi_+(0)$ and $\psi_-(0)$, so it is reasonable to expect them to be slowly varying if ω_1 is small.

Problem 4 of Problem Set 19, was to use the ansatz. You found:

$$\begin{pmatrix} e^{\frac{i}{2}\omega_0 t}\, i \frac{d}{dt} c_+(t) \\ e^{-\frac{i}{2}\omega_0 t}\, i \frac{d}{dt} c_-(t) \end{pmatrix} = -\frac{1}{2}\,\omega_1 \cos \omega t \begin{pmatrix} 0 & e^{-\frac{i}{2}\omega_0 t} \\ e^{\frac{i}{2}\omega_0 t} & 0 \end{pmatrix} \begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix}$$

I didn't have you use the matrix and column vector notation when doing Problem Set 19. Perhaps it is a little more intelligible as two separate equations:

$$\frac{d}{dt} c_+(t) = \frac{i}{2}\,\omega_1 \cos \omega t \cdot e^{-i\,\omega_0 t}\, c_-(t)$$

$$\frac{d}{dt} c_-(t) = \frac{i}{2}\,\omega_1 \cos \omega t \cdot e^{-i\,\omega_0 t}\, c_+(t)$$

The Approximation

The next step is to use $\cos \omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$. Then the two equations are:

$$\frac{d}{dt} c_+(t) = \frac{i}{4}\,\omega_1 \left(e^{i\,\omega t} + e^{-i\,\omega t}\right) \cdot e^{-i\,\omega_0 t}\, c_-(t)$$

$$\frac{d}{dt} c_-(t) = \frac{i}{4}\,\omega_1 \left(e^{i\,\omega t} + e^{-i\,\omega t}\right) \cdot e^{i\,\omega_0 t}\, c_+(t)$$

The approximation is to notice that if ω is near ω_0, which is to say that ω is tuned to be near resonance, then $e^{i\,\omega\,t}\,e^{-i\,\omega_0\,t}$ is slowly varying, but more importantly $e^{-i\,\omega\,t}\,e^{-i\,\omega_0\,t}$ is oscillating at approximately $e^{-2\,i\,\omega_0\,t}$. Rabi says $e^{-2\,i\,\omega_0\,t}\,c_-(t)$ averages to zero! So he simplifies the first equation to:

$$\frac{d}{dt}\,c_+(t) = \frac{i}{4}\,\omega_1\,e^{i\,\omega\,t}\,e^{-i\,\omega_0\,t}\,c_-(t) = \frac{i}{4}\,\omega_1\,e^{i(\omega-\omega_0)\,t}\,c_-(t)$$

Similarly, he simplifies the second equation to:

$$\frac{d}{dt}\,c_-(t) = \frac{i}{4}\,\omega_1\,e^{i(\omega_0-\omega)\,t}\,c_-(t)$$

These equations can be solved! But I am not going to do the general case for you. Instead, I'll just solve them in the case of exact resonance, $\omega = \omega_0$.

The Solution

Using the ansatz and the approximation above, we got ourselves down to two equations. They are coupled differential equations. The derivative of $c_+(t)$ depends on $c_-(t)$ and the derivative of $c_-(t)$ depends on $c_+(t)$. You can take a time derivative of either equation and then use the original equations to simplify. I'm just going to do this for $\omega = \omega_0$, in which case the two equations are just:

$$\frac{d}{dt}\,c_+(t) = \frac{i}{4}\,\omega_1\,c_-(t)$$

$$\frac{d}{dt}\,c_-(t) = \frac{i}{4}\,\omega_1\,c_+(t)$$

Take a time derivative of the first equation and get:

$$\frac{d^2}{d\,t^2}\,c_+(t) = \frac{i}{4}\,\omega_1\,\frac{d}{dt}\,c_-(t)$$

Use the second equation and get:

$$\frac{d^2}{d\,t^2}\,c_+(t) = \frac{i}{4}\,\omega_1\,\frac{i}{4}\,\omega_1\,c_+(t) = -\left(\frac{\omega_1}{4}\right)^2\,c_+(t)$$

That's an easy equation! Its solutions are

$$c_+(t) = A\,\sin\,\frac{\omega_1\,t}{4} + B\,\cos\,\frac{\omega_1\,t}{4}.$$

Let's just use the cosine solution. Putting $B \cos \frac{\omega_1 t}{4}$ back into $\frac{d}{dt} c_-(t) = \frac{i}{4}\, \omega_1\, c_+(t)$ we find that

$$\frac{d}{dt} c_-(t) = \frac{i}{4}\, \omega_1\, B \cos \frac{\omega_1 t}{4}$$

which has the solution.

$$c_-(t) = i\, B \sin \frac{\omega_1 t}{4}$$

If B is 1, all the spins are now flipping back and forth between $+z$ and $-z$! If B is less than 1, we still have whatever B is times all the protons in the tissue's fat and water (which is of course typically an Avogadro's number of protons) flipping back and forth.

Summary

We found $c_+(t) = B \cos \frac{\omega_1 t}{4}$ and $c_-(t) = i\, B \sin \frac{\omega_1 t}{4}$. If B is 1, all the spins are now flipping back and forth between $+z$ and $-z$! If B is less than 1, we still have whatever B is times all the protons in the tissue's fat and water (which is of course typically an Avogadro's number of protons) flipping back and forth. This signal, the combined flipping of many protons, is what is captured in the radio receiver in an MRI machine.

Some Unrelated History

We have seen an application of quantum mechanics that uses the spin-1/2 formalism you studied half a semester ago, and the nuclear physics you learned in the last few weeks. We followed the analysis made easy for us by I. I. Rabi, who won the Nobel Prize in Physics in 1944 for the discovery of nuclear magnetic resonance, and for which huge numbers of injured and sick people have benefitted from.

When Rabi was asked, upon returning to Los Alamos 40 years after working there, what he felt, he said "sorrow, that the place still exists." See https://youtu.be/Lm412GwSkIM. From a few minutes after the first atomic bomb test off Rabi was moved to devote himself to arms control. Be careful not to assume that anyone who worked on the bomb, while regretting that atomic and thermonuclear bombs still exist in vast quantities and threatens u continuously, believes that it should not have been built under the conditions that the Allies faced at the time. That would be simplistic revisionism. For a pensive and carefully thought out statement from Oppenheimer himself, see https://youtu.be/AdtLxlttrHg.