Schrödinger's Equation

In Chapter 10, you are already studying the solution of Schrodinger's Equation even though Moore is not going to explain it until Chapter 12. Moore often does this (shows you a solution to help you gain intuition before even giving you the full problem).

I'd like to introduce Schrodinger's Equation now, and in fact, we have already been doing it, so this is mostly a summary of what you know so far, but at the end I go somewhat further.

De Broglie Waves

Let's recall de Broglie's explanation for electron interference. He supposed that an electron was a wave and its wavefunction was

$$e^{ipx/\hbar - iEt/\hbar}$$

In this wave function, p is the electron's momentum and E is its energy. We know that $E = \frac{p^2}{2m}$ for a free particle.

The Time-Dependent Schrödinger Equation

We can capture the equation

$$E = \frac{{p_x}^2}{2m} + V(x, t)$$

by looking at what we would have to do to de Broglie's waves.

To get E, which you are supposed to pretend for a moment that you don't know, you could apply $i\hbar \frac{\partial}{\partial t}$ to de Broglie's wave.

To get p_x which you must also pretend for a moment that you don't know, you could apply $\frac{\hbar}{i} \frac{\partial}{\partial x}$ to de Broglie's wave. But we want $\frac{{p_x}^2}{2m}$ so we have to apply $\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$ to the wave. Now that we have figured out what to apply, we can do it to any wavefunction $\psi(x, t)$, not just to de Broglie's plane waves.

In other words, we could make the leap that energy equals kinetic energy plus potential energy implies that:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x, t)\, \psi(x, t)$$

We have Schrödinger's equation, and shortly you will see that there is a second version of Schrödinger's equation.

The Ansatz and the Time-Independent Schrödinger Equation

We know that solutions of different energies exist (otherwise we wouldn't have stable atoms), so we guess that

$\psi(x, t) = e^{-i E_n t/\hbar} \psi(x)$ <<<=== the ansatz

where E_n is one of these energies. We plug this guess into

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x, t)\, \psi(x, t)$$

and we get

$$E_n e^{-i E_n t/\hbar} \psi(x) = \frac{-\hbar^2}{2m} e^{-i E_n t/\hbar} \frac{d^2 \psi(x)}{dx^2} + V(x, t)\, e^{-i E_n t/\hbar} \psi(x)$$

PROVIDED that $V(x, t)$ is in fact time-independent, e.g., $V(x, t) = V(x)$, then all the time-dependence in the equation is the same in every term (just $e^{-i E_n t/\hbar}$) and we can cancel it off, leaving:

$$E_n \psi(x) = \frac{-\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x)\, \psi(x)$$

Because we have two versions of Schrödinger's equation, we have to give them different names. This one is the "time-independent Schrödinger equation" and the one we had in the previous sub-section is the "time-dependent Schrödinger equation."

Combining Solutions

If we have two solutions with energies E_n and E_m, we can combine them as follows:

$$\psi(x, t) = \frac{a_n e^{-iE_n t/\hbar} \psi_n(x) + a_m e^{-iE_m t/\hbar} \psi_m(x)}{\sqrt{|a_n|^2 + |a_m|^2}}$$

where a_n and a_m are any constants, and the $\sqrt{|a_n|^2 + |a_m|^2}$ in the denominator is there to keep the combination normalized. The important thing about this combination is that it is also a solution of the time-dependent Schrödinger's equation!