Them’s the Rules

Wherein the rules of quantum mechanics are given, using the two spin states of the electron as the example, and ignoring the electron’s position and momentum.

Recap of and Supplement to Last Handout

In the last handout, we listed the properties of the electron: its position, its momentum, and its spin, $\vec{S}$. It also has a magnetic moment, but the magnetic moment is always directly proportional to its spin,

$$\vec{\mu} = g \frac{-e}{2m} \vec{S}$$

so that isn’t a separate property. If the electron is in a magnetic field, $\vec{B}$, it experiences a torque:

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

If it is in a non-uniform magnetic field, it also experiences a force that deflects the beam:

$$F_i = \frac{\partial}{\partial x_i} \vec{\mu} \cdot \vec{B}$$

$\vec{B}$ points whatever way a compass would point and the length of $\vec{B}$ is proportional to the strength of the torque, and that is all you have to know about $\vec{B}$. I didn’t give you the formulas for $\vec{\tau}$ and F_i in the last handout. When I presented the material, I only waved my hands and tried to give you a feel for what a spinning ball filled with charge might do in a magnetic field. During the hand-waving. I appealed to what a magnet would do and I claimed that a spinning ball filled with charge behaves like a magnet. I got out a gyroscope and tried to show everyone that torque quite counterintuitively causes precession.

Some Relevant Special Cases

Torque in the special case that $\vec{B}$ only has a component B_z

We can often choose our coordinates so that we have the special case where $\vec{B}$ only has a component B_z. Then $|\vec{\tau}| = |\vec{\mu}|\,|\vec{B}| \sin\theta_\mu$, and the torque is such that it is trying to align the south pole of the spinning ball toward the north pole of whatever magnet it is near (or vice versa). In Moore Problem

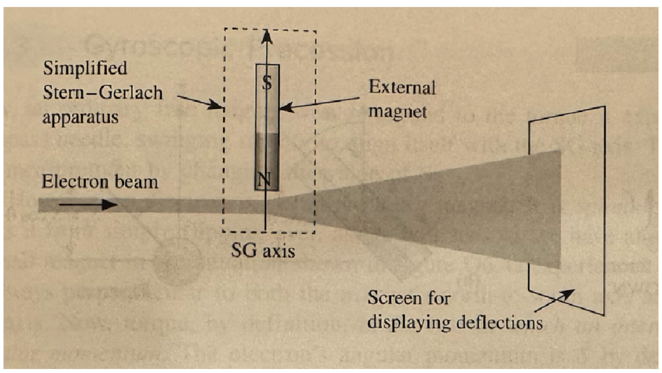
Q6M.4 you explored the consequences of this formula for torque and found the period of the precession. For convenience in Q6M.4, Moore let $b = |\vec{\mu}|\,|\vec{B}|$.

Force in the special case where B_z is getting weaker as you move in the -z direction

In the special case that B_z is only getting weaker (or stronger) as you move in the -z (or $+z$) direction we only have an F_z and it is given by $F_z = |\vec{\mu}|\,\frac{\partial B_z}{\partial z}\cos\theta_\mu$.

Simplified Stern-Gerlach Apparatus

Here is Fig. Q6.2 from p. 89, and you start to see why these special cases might be relevant:



Being Careful with the Minus Signs (for the Sticklers!)

One of the reasons I just waved my hands last time is that there is a bonanza of minus signs to consider, and I realized while presenting that I was going to have to sit somewhere quiet to get them all right. This sub-section is only for those sticklers who also want to get the minus signs right. The upshot is that as diagrammed, spin up deflects up, and spin down deflects down.

Let's take the upward-pointing arrow in the diagram above to be the +z direction. The north pole of the magnet Moore has drawn is pointing down at the beam, so B_z is pointing down, and therefore B_z is negative. Also, the magnetic field is getting weaker as it gets further from the north pole of the magnet so $\frac{\partial B_z}{\partial z}$ is negative. But θ_μ is the magnetic moment direction, not the spin direction, so if you want to rewrite the force formula in terms of the spin and the spin direction, θ, it is

$$F_z = |\vec{\mu}|\,\frac{\partial B_z}{\partial z}\cos\theta_\mu = -|\vec{\mu}|\,\frac{\partial B_z}{\partial z}\cos\theta = -g\,\frac{e}{2m}\,|\vec{S}|\,\frac{\partial B_z}{\partial z}\cos\theta = g\,\frac{e}{2m}\,|\vec{S}|\,\left|\frac{\partial B_z}{\partial z}\right|\cos\theta$$

So in Moore's diagram, when an electron passes through the apparatus with its spin pointing upward (any θ less than 90°), it is deflected upward, and when its spin is pointing downward (any θ greater than 90°), it is deflected downward.

Let’s Focus on Two States

To create some mathematics that describes the incredibly important example of an electron passing through the Stern-Gerlach apparatus, (i) ***for a time we will ignore the position and momentum of the electron.*** Also, we acknowledge that, (ii) whenever the electron goes through a Stern-Gerlach apparatus with its axis in the *z*-direction, that the spin measured to be either $S_z = +\frac{1}{2}\hbar$ or $S_z = -\frac{1}{2}\hbar$.

We capture the fact that there are two states, but we don’t know what they are, by giving them symbols in the bra-c-ket notation: |+z> and |-z>. If the Stern-Gerlach apparatus had been oriented some other direction, like the *x*-direction, then again experiment shows that there are two states, and we give them the symbols: |+x> and |-x>. Part of what we are positing is that although we don’t know the relationship, these two new states are related to the |+z> and |-z> states. They aren’t entirely new and different.

Ok, so we have no idea what the relationship is between the two pairs we have listed, nor for that matter, between either of the above pairs and |+θ> and |-θ>, where now we are being more general, and $\theta = 0$ is for the states coming out of the *z*-direction Stern-Gerlach apparatus, $\theta = 90^\circ$ is for the states coming out of the *x*-direction Stern-Gerlach apparatus, and all the values of θ for Stern-Gerlach apparatuses oriented from 0° to 180° are now allowed. We could even contemplate the third axis, with the magnetic field oriented along the beam line |+y> and |-y> but let’s go easy on ourselves and save that for a bit later.

Superpositions of States

A big part of the intellectual leap that you must take, and that was made in the 1920s, is that electrons can be in superpositions of states. ***Considering the*** |+z> ***and*** |-z> ***states as a “basis” we posit that any state of the electron*** |ψ> ***can be described as***

$$|\psi\rangle = c_+ |+z\rangle + c_- |-z\rangle$$

where c_+ and c_- are complex numbers and are called “probability amplitudes.”

If an electron enters the Stern-Gerlach apparatus with the *z*-direction orientation in the above superposition, we posit that the ***probability*** of it being in the |+z> state and being found among the electrons deflected into the upper patch will be

$$P_+ = |c_+|^2$$

and the ***probability*** of the electron being in the |-z> state and being found among the electrons deflected into the lower patch will be

$$P_- = |c_-|^2$$

In those formulas, that isn't the ordinary absolute value, because these are complex numbers — it is the complex version of modulus that you learned about in Churchill, Brown, and Verhey. And of course, since it has to be one or the other then:

$$P_+ + P_- = |c_+|^2 + |c_-|^2 = 1$$

Them's the Rules

These are simple and yet fantastic assertions. Why should it be possible to add states? Why should the coefficients in the combinations be allowed to be complex? Why should the |+x> and |-x> states be linear combinations of the |+z> and |-z> states? Why should the square of the complex modulus be the probability of an experimental measurement? And why should we be able to get away with studying the spin of the electron while ignoring its position and momentum!?

Accept the leaps of Heisenberg, Dirac, and Schrödinger. See where they lead.

Also, now you can see why the two-state system of the electron is so important for learning the rules. If we first found and studied a system with only one state, we would be bored and learn next to nothing about the rules. If instead, we also added into our study of electrons their position and momentum, either of which has an infinity of vector values, then we would be overwhelmed by the infinity of states and the rules.

In the sequences, 1, 2, ∞, or Papa Bear, Mama Bear, and Baby Bear, one of the choices is just right. An infinity of states is too hard! One state is too easy! Two is just right! So we will study two-state systems until you know them backwards and forwards. Then in Q9 Moore will re-introduce position and momentum and an infinity of possible states.

Summary

Q7 is a very important chapter and the above is my last ditch attempt to prep you for it. Go forth and conquer.