Brian’s Lightning Calculus Refresher

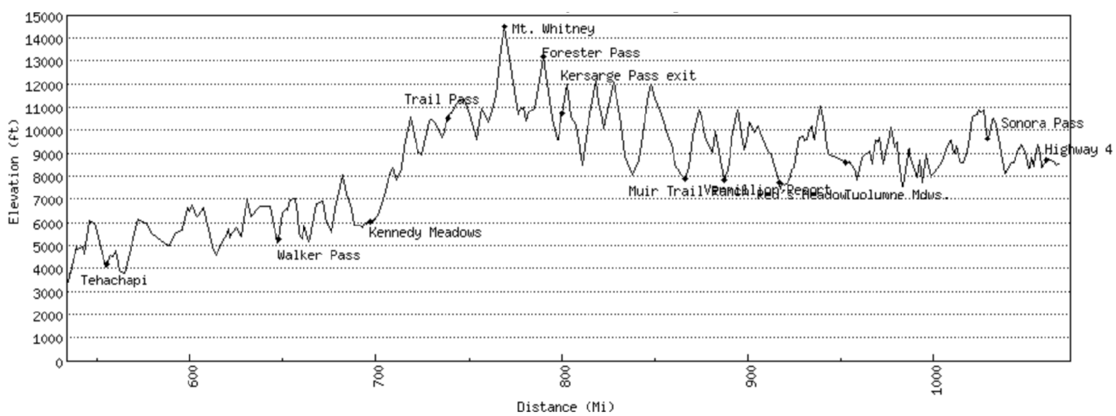
Revision 1.1 — As we encounter more calculus that we need and if there are typos or points lacking clarity, this document will be revised and augmented.

Functions and Variables — in Math

In advanced high-school and college mathematics, x is almost always the independent variable, and functions are written as $y = f(x)$.

When functions are graphed, it is conventional to make the x-axis the horizontal axis and the y-axis the vertical axis.

An example would be the height of the Pacific Crest Trail as you travel northward.



In this chart, on the x-axis, is the distance (in miles) along the trail measured from Campo, California. On the y-axis, is the elevation of the trail (in feet) above sea level.

Functions and Variables — in Physics

In physics, time is usually the independent variable. All sorts of things can depend on time in physics, but the first and most common thing is the position of a point particle. In 3-d, the position of a point particle requires three numbers. We usually set up a coordinate system with axes labeled x, y, and z.

So the position of a particle at any time t, is specified by three functions

$x = f(t)$
$y = g(t)$
$z = h(t)$

A subtle distinction — a function vs. the value of a function

This might be too subtle to mention.... Physicists are terse. We don't generally bother making any notational distinction between the function f and the value of that function at some time $x = f(t)$.

So we just write:

x(t)
y(t)
z(t)

Vectors

Physicists typically assemble these three functions or values into a vector which we write as

$$\vec{r}(t) = (x(t),\ y(t),\ z(t))$$

The little arrow over the r is there to remind you that this is actually three numbers!

Bringing it Down to 1-D

Let's imagine that a particle is confined to one dimension. An example of a "particle" would be a train which is confined to a track. You can say where the "particle" is by specifying where the locomotive is. This just needs one number since the train can't leave the tracks.

At any given time t, we can give the coordinate, which we might call $x(t)$. The coordinate might be how far north of Reno the train has gotten, and if the train is south of Reno, it would have negative values.

Average Slope in Math

Since x is usually the independent variable and $y = f(x)$ is usually the function in mathematics, we can calculate the average slope which is by definition the rise over the run, and here is the formula:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Make a diagram for yourself that shows why the above is the rise over the run and why it is the average slope between x_1 and x_2. Label everything!

Average Velocity in Physics

As already noted, x is usually the dependent variable t is usually the independent variable in physics. So the slope on the x vs. t graph is:

$$\frac{x_2-x_1}{t_2-t_1}$$

But this is coordinate change over time change, so this is average velocity! Note that it can be positive or negative, unlike speed, which is never negative.

Derivative in Math

The derivative is just the slope at a single point instead of the average slope. How do you calculate it?!? First, some notation. We will let the point we are focused on be x instead of x_1, and we will let $x_2 = x + \Delta x$.

Make those substitutions and convince yourself that the average slope in math becomes:

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Now we imagine that Δx gets tinier and tinier. This is written as follows:

$$\text{lim}_{\Delta x\to 0}\ \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

I had a terrible time typesetting that. The $\Delta x \to 0$ is usually placed directly under the “lim.”

This combination occurs so much that it has a name “the derivative of f at x” and its own symbol:

$$\frac{df}{dx} \equiv \text{lim}_{\Delta x\to 0}\ \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The triple = sign means “definition.”

Velocity in Physics

Instead of the independent variable being x it is t. So we have:

$$\frac{df}{dt} \equiv \text{lim}_{\Delta t\to 0}\ \frac{f(t+\Delta t)-f(t)}{\Delta t}$$

Also, as noted above, physicists are terse, so they don't bother with giving the function its own name. They just write the right-hand-side as:

$$\lim_{\Delta t\to 0} \frac{x(t+\Delta t)-x(t)}{\Delta t}$$

This comes up so often, it also has a notation:

$$\frac{dx}{dt} \equiv \lim_{\Delta t\to 0} \frac{x(t+\Delta t)-x(t)}{\Delta t}$$

Before we took the limit that Δt was tinier and tinier (the official nomenclature is "infinitesimal"), this was the average velocity.

Is this not now something worth calling the "instantaneous velocity!?!" We write:

$$v \equiv \frac{dx}{dt} \equiv \lim_{\Delta t\to 0} \frac{x(t+\Delta t)-x(t)}{\Delta t}$$

3-D Velocity

All we have to do is all of this in triplicate in 3-d. The combinations occur so much that they have common notations:

$$v_x \equiv \frac{dx}{dt} \equiv \lim_{\Delta t\to 0} \frac{x(t+\Delta t)-x(t)}{\Delta t}$$

$$v_y \equiv \frac{dy}{dt} \equiv \lim_{\Delta t\to 0} \frac{y(t+\Delta t)-y(t)}{\Delta t}$$

$$v_z \equiv \frac{dz}{dt} \equiv \lim_{\Delta t\to 0} \frac{z(t+\Delta t)-z(t)}{\Delta t}$$

And we can assemble v_x, v_y, and v_z into a vector

$$\vec{v} = (v_x, v_y, v_z)$$

and we can also write

$$\vec{v} = \frac{d\vec{r}}{dt}$$

This is probably what Moore was referring to that he said we could delay for a while. This is a bit of vector calculus. We are taking derivatives of vectors and getting vectors!

Common Derivatives in Math

You usually memorize common derivatives. For example for the function $f(x) = x^n$,

$$\frac{d\,x^n}{dx} = n\,x^{n-1}$$

Some other popular ones are

$$\frac{d \ln x}{dx} = \frac{1}{x}$$

$$\frac{d\,e^x}{dx} = e^x$$

$$\frac{d \sin x}{\mathrm{dx}} = \cos x$$

$$\frac{d \cos x}{\mathrm{dx}} = -\sin x$$

A few other rules that are super-handy are product rule and chain rule. Here is product rule:

$$\frac{d\,f \cdot g}{\mathrm{dx}} = \frac{d\,f}{dx} \cdot g + f \cdot \frac{d\,g}{dx}$$

Here is chain rule. Create a new function as follows: $h(x) = f(g(x))$. This is called function composition and can be read "*h* is *f* composed with *g*." Then

$$\frac{dh}{dx} = \frac{df}{dg}\,\frac{dg}{dx}$$

Yikes!

If that was on the hairy edge, I understand.

A simple example of chain rule is when the function g is just multiplication by a constant that we can call a. Then $h(x) = f(a\,x)$. The result of chain rule in this case is

$\frac{\mathrm{dh}}{\mathrm{dx}}$ = the derivative of f evaluated at $a\,x$ times a

I wrote it out in words because precise notation for function composition gets a bit sticky and is a place where the terseness of physicists can get them into trouble.

Common Derivatives in Physics

Here are three for you to evaluate:

1. $x(t) = v_0\, t$

2. $x(t) = \frac{1}{2}\, a\, t^2$

3. $x(t) = A \sin \omega t$

TWO NOTES ABOUT THE LAST ONE: (1) I don’t use gratuitous parenthesis. You were probably taught to write $x(t) = A \sin(\omega t)$ with the extra parenthesis. Sometime I will explain when it is allowed to omit them. Again, you will find physicists to be terse. (2) You will need chain rule and my simple example of function composition above to take the derivative of $\sin \omega t$ because this is the sine of a product, not just the sine.

In the above equations, v_0, a, A, and ω are all constants.

In all three cases, please take the time to calculate $v \equiv \frac{dx}{dt}$, or none of this is going to stick.