Brian's Lightning Trig Refresher

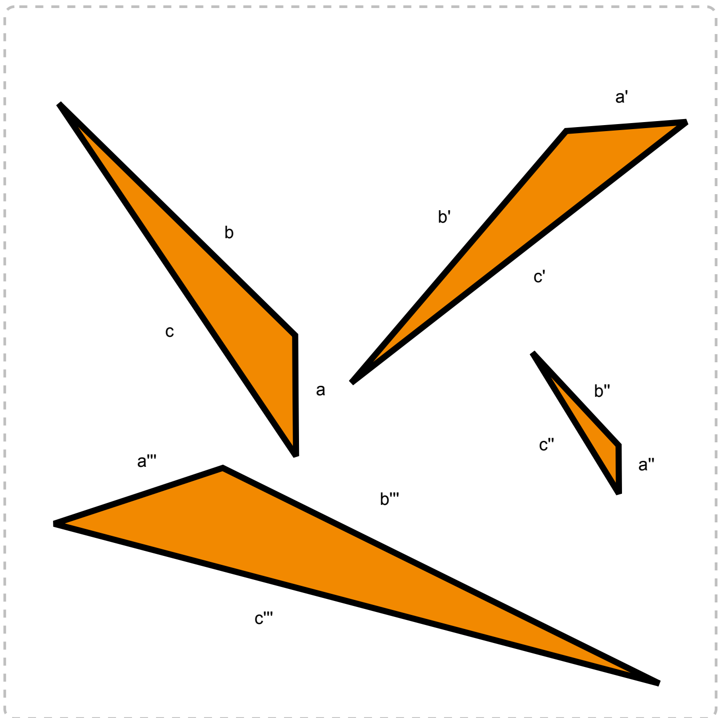
Revision 1.0 — If we encounter more trig that we need and/or if there are typos or points lacking clarity, this document will be revised and augmented.

Similar Triangles

If two triangles have the same interior angles, then they are similar.

What does that mean? They might be rotations of each other. They might be reflections of each other. Or they might be rescalings of each other.

These are all things you can do on a photocopier. You can rotate the paper. You can flip the paper over (and imagine that you can still see through it). Or you can press the reduce or enlarge button. These triangles are all similar:



The powerful fact about similar triangles is that if the lengths of the three sides of one are a, b, and c and the lengths of the corresponding sides of another are a', b', and c' then

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}$$

The above expresses that no matter what you do at the photocopier the new triangle's dimensions *are in a single fixed ratio* to the old triangle's dimensions.

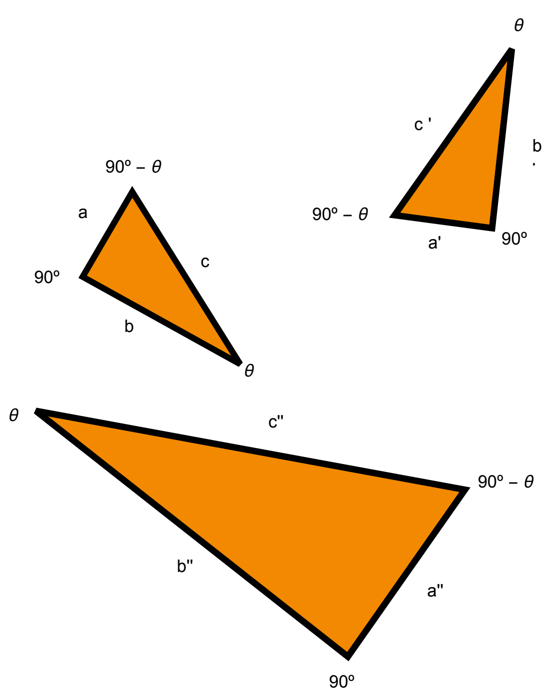
There are so many ways of rearranging those equations, you wouldn't want to list all of them. How about these as three examples:

$$\frac{a}{c} = \frac{a'}{c'} \quad \text{and} \quad \frac{b}{c} = \frac{b'}{c'} \quad \text{and} \quad \frac{a}{b} = \frac{a'}{b'}$$

Right Triangles and Similarity

If you name the angles of a triangle to be α, β, and γ then the equation $\alpha + \beta + \gamma = 180^\circ$ is always true no matter what the triangle. If you further specify that one of these angles is 90°, then things simplify. Let's have γ be the angle that is 90°. Then $\alpha + \beta + 90^\circ = 180^\circ$ or $\alpha + \beta = 90^\circ$ or $\beta = 90^\circ - \alpha$. Oh my. That means there is only one angle to specify in a right triangle! One is by assumption 90°. Another is the one I called α. The third β is $90^\circ - \alpha$.

That means that all right triangles with the same angle α are similar because once you have specified that angle the others are specified!! We usually call the one angle that we have to specify θ, and I am going to put that angle opposite side a. In the picture below, all these right triangles are similar because they all have one angle that is θ:



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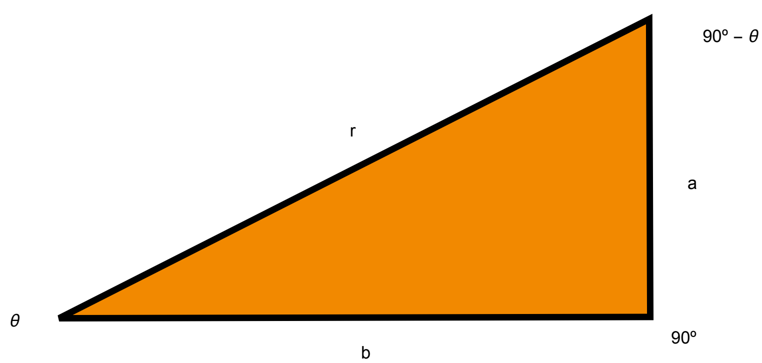
The Trig Functions

If you have agreed to everything so far, then you know that for any two right triangles with one angle being θ, that

$$\frac{a}{c} = \frac{a'}{c'} \quad \text{and} \quad \frac{b}{c} = \frac{b'}{c'} \quad \text{and} \quad \frac{a}{b} = \frac{a'}{b'}$$

Since the only thing these three ratios depend on is θ, we can make give the ratios names and make tables of what their values. The name for $\frac{a}{c}$ is sine of θ, the name for $\frac{b}{c}$ is cosine of θ, and the name for $\frac{a}{b}$ is tangent of θ. As functions, we we use the abbreviations $\frac{a}{c} = \sin\theta$, $\frac{b}{c} = \cos\theta$ and $\frac{a}{b} = \tan\theta$.

People usually remember this as SOHCAHTOA, and that works, but it is more common to need to use the trig functions as follows:



All I did was rename the long side r instead of c, and then the best way to remember the three trig functions $\frac{a}{c} = \sin\theta$, $\frac{b}{c} = \cos\theta$ and $\frac{a}{b} = \tan\theta$, is

$$a = r\sin\theta, \quad b = r\cos\theta, \quad \text{and} \quad \frac{a}{b} = \tan\theta$$

because that is the way you most often will want to write use the formulas.

Trig Identities

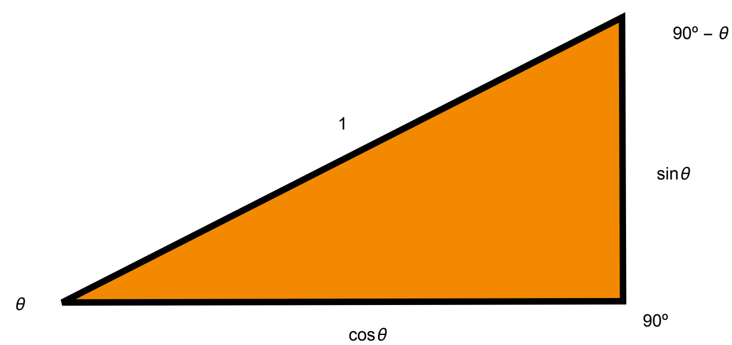
Because we are dealing with right triangles, we have the Pythagorean theorem: $a^2 + b^2 = r^2$. Putting in $a = r\sin\theta$, and $b = r\cos\theta$ we learn:

$$r^2 \sin^2\theta + r^2 \cos^2\theta = r^2$$

Cancelling the r^2 we have

$$\sin^2\theta + \cos^2\theta = 1$$

A different way of seeing the same thing (rather than cancelling the r^2) is just to use the special triangle with length 1 for r:



Then the Pythagorean theorem immediately gives $\sin^2\theta + \cos^2\theta = 1$.

For some more trig identities, we observe that we could have made $90\,^\circ - \theta$ the independent angle, we see that a relationship between cosine and sine:

$$\cos\theta = \sin(90\,^\circ - \theta) \quad \text{and} \quad \sin\theta = \cos(90\,^\circ - \theta)$$

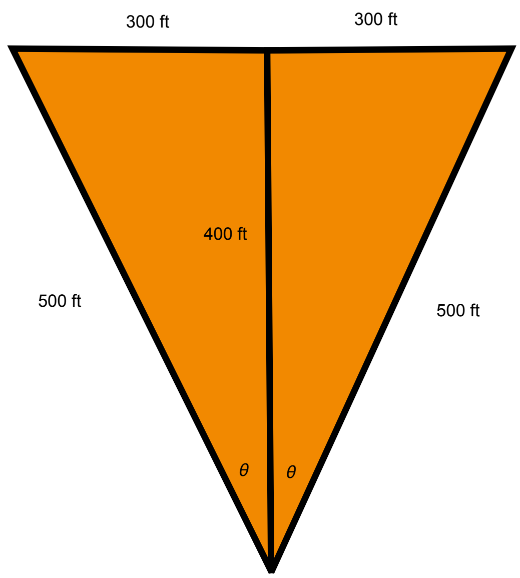
As another relationship, because $\frac{a}{b} = \frac{a/r}{b/r}$ we learn that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

You can get other stinking useful trig identities by stacking two triangles. You end up having one triangle with angle α, another with angle β, and the stacked triangle has angle $\alpha + \beta$, so maybe it isn't a surprise that there are identities for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ that involve $\sin\alpha$, $\cos\alpha$, $\sin\beta$, and $\cos\beta$.

Pythagorean Triples

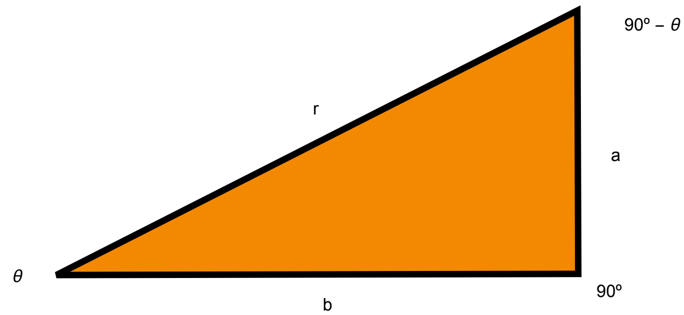
To make good problems, physicists often use the famous 3-4-5 and 5-12-13 right triangles. For example, they will say a canyon is six hundred feet wide and four hundred feet deep. Well then, half the canyon width is three hundred feet. This is setting you up (quite possibly) to notice that the diagonals in this picture are 500 feet long:



If you labeled the angle θ and then got out your calculator and did $\theta = \tan^{-1} \frac{300\text{ ft}}{400\text{ ft}}$ and then later on, you needed to do something like find the cosine of θ, you could have skipped getting out the calculator. You can read the cosine of theta right off the figure. It is $\frac{b}{r} = \frac{400\text{ ft}}{500\text{ ft}} = \frac{4}{5}$.

Application to Work

We got kind of balled up in class trying to do the work of gravity on the car on the incline. This (in addition to it being way overdue) is what lit a fire under me to get the trig refresher out. Take a look at this picture. If you saw this picture after reading this refresher, and you needed to know the combination $r\cos\theta$, can't you just read it off without any funky cosine-of-the-arctangent steps?



What is rcosθ in terms of the things that are labeled? What is $r\cos(90\,^{\circ} - \theta)$ in terms of the things that are labeled?

Once you see things like that, boom, you will not have to do much work to see that for a car going up an incline, the work done by gravity on the car is $W = \vec{F} \cdot \Delta\vec{r} = -mg\Delta h$.

You still need to do a little work to draw out all the angles and get the minus sign right, but you don't have to invoke the arctangent. It might be slightly easier to see how the signs work out if you do a car going down an incline. Then the work done by gravity will be positive.

It is because the angle between $\vec{F}$ and $\Delta\,\vec{r}$ is greater than 90° for a car going uphill that a minus sign shows up in the work.