
Another Fourier Problem

A Fourier Localization Problem with no Ringing

For Quantum Mechanics, Monday, Mar. 16, 2026

The Function to Approximate With Cosines

An issue with our first and easiest example of localization (a step-up-and-step-down function that was nonzero only between $-\frac{L}{8}$ and $\frac{L}{8}$) appeared at the discontinuities.

It is difficult for a bunch of continuous functions (in this case the sum of any number of cosines) to well-approximate any function with a discontinuity, even if the function is even and vanishes at the edge of the box.

The function when approximated by cosines exhibits ringing.

Here and below, I will frequently specialize to the case $L = 4$ for definiteness. The box then goes from -2 to 2 .

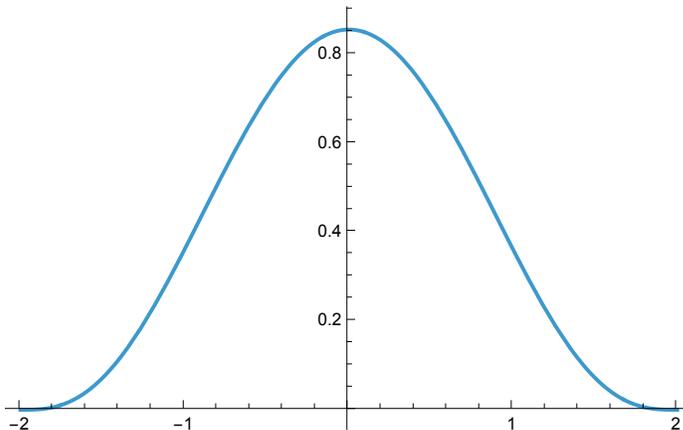
Here is a polynomial of degree six, that is concentrated on $[-1, 1]$, and small on $[-2, -1]$ and on $[1, 2]$:

In[]:=

$$\psi[x_] := \frac{\sqrt{3003}}{64} \left(1 - \frac{x^2}{4}\right)^3$$

`Plot[ψ[x], {x, -2, 2}]`

Out[]:=

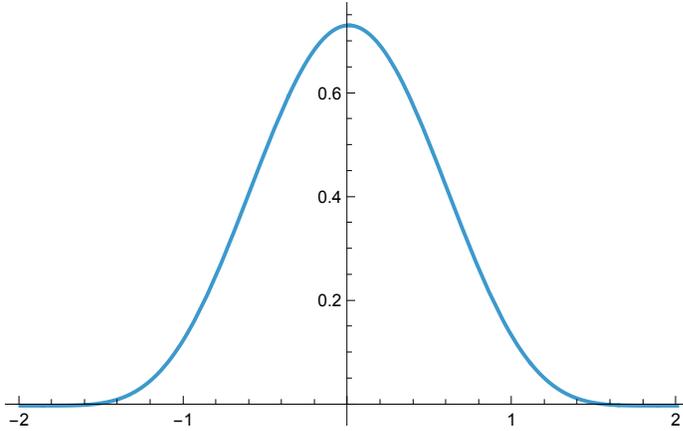


Square this function to get a probability distribution:

```
In[ ]:= prob[x_] :=  $\psi[x]^2$ 
```

```
Plot[prob[x], {x, -2, 2}]
```

```
Out[ ]:=
```



Confirm that the constant out front has been chosen so that the probability of being somewhere in the box is 1:

```
In[ ]:= Integrate[prob[x], {x, -2, 2}]
```

```
Out[ ]:=
```

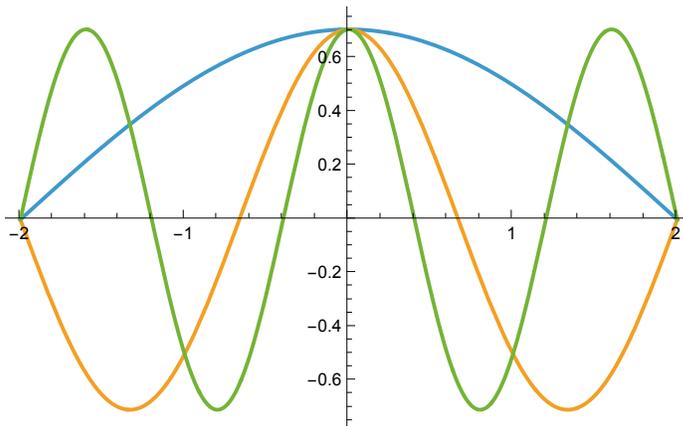
```
1
```

YOUR TURN — Compute the Coefficients

Again, the only cosines we have to consider are $\sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L}$ where n is any odd integer. Here are the first three of those functions:

```
In[ ]:= Plot[{c[x, 1], c[x, 3], c[x, 5]}, {x, -2, 2}]
```

```
Out[ ]:=
```



Apply the powerful mathematics of Fourier analysis for $n = 1, 3, 5$:

```
In[*]:= L = 4;
```

```
c[x_, n_] := Sqrt[2/L] * Cos[n Pi x / L];
```

```
tableOfCoefficients = Table[Integrate[psi[x] * c[x, n], {x, -L/2, L/2}], {n, 1, 5, 2}];
```

Find out what the coefficients are numerically. To check your work, I will show you the third coefficient (for $n = 5$):

```
In[*]:= N[tableOfCoefficients][[3]]
```

```
Out[*]=
```

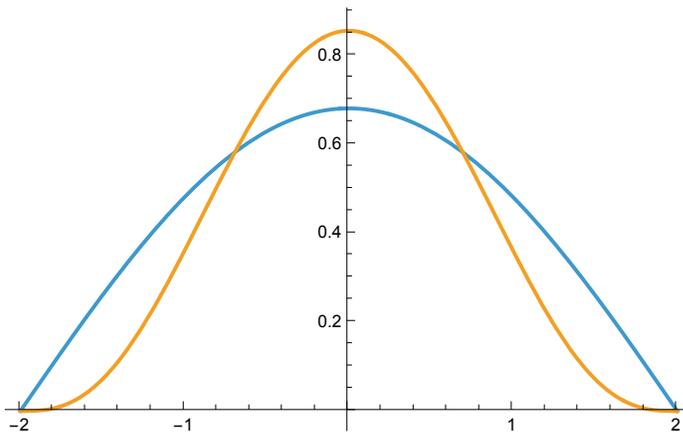
```
-0.0223933
```

Assembling the Series — First Term Only

```
oneTermOnly[x_] := tableOfCoefficients[[1]] * c[x, 1]
```

```
Plot[{oneTermOnly[x], psi[x]}, {x, -2, 2}]
```

```
Out[*]=
```



YOUR TURN AGAIN — First Two Terms

Plot the result with the first two terms.

YOUR TURN AGAIN — First Three Terms

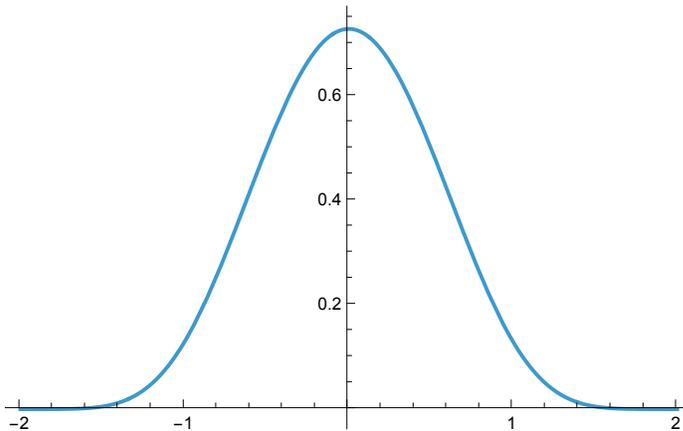
Plot the result with the first two terms.

YOUR TURN AGAIN — Probability — Square of the First Three Terms

Let's square it and see what it looks like, since that gives the probability distribution. Plot the square of the first three terms. Make sure your result looks like mine:

```
In[ ]:= Plot[(threeTerms[x])2, {x, -2, 2}]
```

```
Out[ ]:=
```



YOUR FINAL CHECK — Normalization — Integrate the Square of the First Three Terms

Double-check that your mathematics agrees with mine by doing the integral that should come out very near 1, and would quickly get even closer to 1 if we included more terms:

```
In[ ]:= N[Integrate[(threeTerms[x])2, {x, -2, 2}]]
```

```
Out[ ]:=
```

```
0.99998
```

Conclusion

Is it not remarkable that out of the following three cosines, we were able to make a polynomial of degree 6 that was reasonably localized?

```
In[ ]:= Plot[{c[x, 1], c[x, 3], c[x, 5]}, {x, -2, 2}]
```

```
Out[ ]:=
```

