Handout and Problems to go with Chapter 7

For Quantum Mechanics, Monday, Feb. 23, 2026

1. The Feynman Problem for Chapter 7

2. A Partial Introduction to Localization and Fourier Analysis

Consider a particle somewhere in a one-dimensional box of length L. We'll have the x coordinate describing the box run from $-\frac{L}{2}$ to $\frac{L}{2}$. Later I will set $L = 4$ for definiteness.

To illustrate localization, we would like to construct a wave function out of sines and cosines that localize the particle within the region $x = \frac{-L}{8}$ to $x = \frac{L}{8}$. If the particle is going to be evenly-likely found within this range, which is $\frac{L}{4}$ long, one way to do that is to have its amplitude be the constant $\sqrt{\frac{4}{L}}$ within this range, and zero elsewhere.

Because I have chosen a localization that is symmetrical about the origin, it turns out we don't need sines and cosines. We only need the cosines because they are symmetric (even) whereas the sines are antisymmetric (odd).

Some general considerations based on boundary conditions are that the only cosines we have to consider are $\sqrt{\frac{2}{L}} \cos \frac{n \pi x}{L}$ where n is any odd integer. The square root, $\sqrt{\frac{2}{L}}$, out front is put there so that the average of the function-squared is $\frac{1}{L}$. If that function were used all by itself as a probability amplitude in the box of length L, and we then squared it to get a probability function, then the probability to find the particle somewhere in the box would be 1.

There is powerful mathematics from the world of Fourier analysis, which you definitely are not meant to understand at the moment, but which I can use to compute the appropriate combination of cosines. I will now apply the procedure, which involves integration, and I am only going to do it for $n = 1, 3, 5, \ldots, 15$. If you want to go past $n = 15$ (it is an infinite series) feel free to go further.

Since we are only doing n odd, $n = 15$ is actually just the first 8 cosines in the series.

In[ ]:=
```
L = 4;
c[x_, n_] := Sqrt[2/L] × Cos[n Pi x/L];
tableOfCoefficients = Table[Integrate[Sqrt[4/L] × c[x, n], {x, -L/8, L/8}], {n, 1, 15, 2}]
```

Out[ ]=

$$\left\{\frac{4\sqrt{2}\,\mathrm{Sin}\left[\frac{\pi}{8}\right]}{\pi}, \frac{2\,\mathrm{Csc}\left[\frac{\pi}{8}\right]}{3\pi}, \frac{2\,\mathrm{Csc}\left[\frac{\pi}{8}\right]}{5\pi}, \frac{4\sqrt{2}\,\mathrm{Sin}\left[\frac{\pi}{8}\right]}{7\pi}, -\frac{4\sqrt{2}\,\mathrm{Sin}\left[\frac{\pi}{8}\right]}{9\pi}, -\frac{2\,\mathrm{Csc}\left[\frac{\pi}{8}\right]}{11\pi}, -\frac{2\,\mathrm{Csc}\left[\frac{\pi}{8}\right]}{13\pi}, -\frac{4\sqrt{2}\,\mathrm{Sin}\left[\frac{\pi}{8}\right]}{15\pi}\right\}$$

Perhaps it is nice, and maybe even more convenient to see what these coefficients are numerically:

In[ ]:=
```
N[tableOfCoefficients]
```

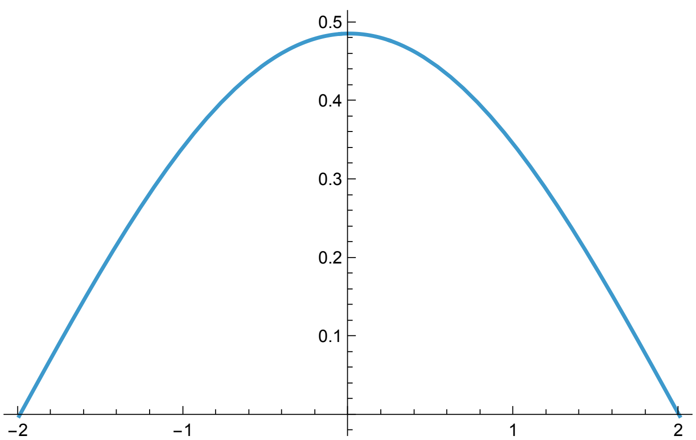
Out[ ]=
```
{0.689072, 0.554523, 0.332714, 0.0984389,
 -0.0765636, -0.151233, -0.127967, -0.0459382}
```

Assembling the Series — First Two Terms

First term only:

In[ ]:=
```
Plot[tableOfCoefficients[[1]] × c[x, 1], {x, -2, 2}]
```

Out[ ]=

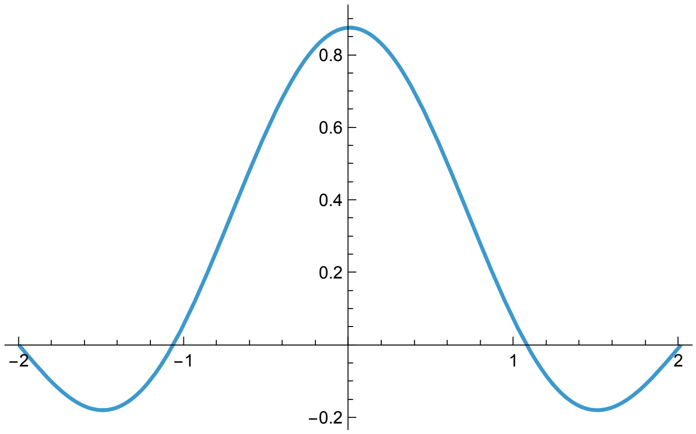


Note that coefficient 2 multiplies the cosine with $n = 3$, coefficient 3 multiplies the cosine with $n = 5$, *etc.*

Here is the localized function being built up two terms:

In[•]:=
```
Plot[tableOfCoefficients⟦1⟧ × c[x, 1] + tableOfCoefficients⟦2⟧ × c[x, 3], {x, -2, 2}]
```

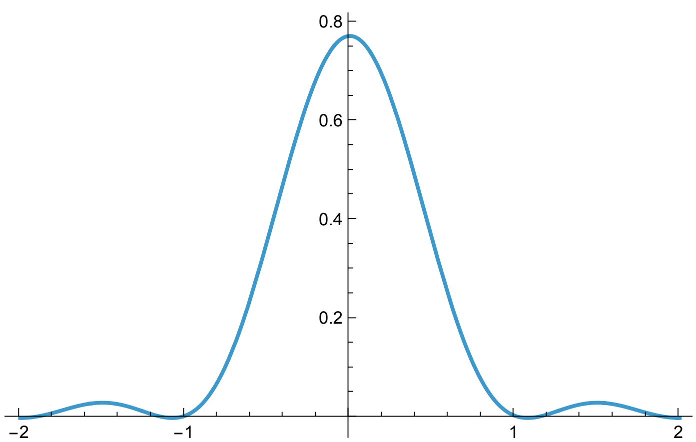
Out[•]=



Let's square it and see what it looks like, since that gives the probability distribution:

In[•]:=
```
Plot[
 (tableOfCoefficients⟦1⟧ × c[x, 1] + tableOfCoefficients⟦2⟧ × c[x, 3])^2, {x, -2, 2}]
```

Out[•]=



That's starting to look localized! Sure it is still smooth, but can you believe that additional terms might make it approach 1 better in the range $-\frac{L}{8}$ to $\frac{L}{8}$, which with $L = 4$ is $-\frac{1}{2}$ to $\frac{1}{2}$, and be zero elsewhere?

FOR YOU — Find some graphing program and do another term or two in the series — graph both the function and its square. The numerical values of the table of coefficients might be easier to use than the exact ones. Also, note that I have chosen $L = 4$ so your graphs will go from −2 to 2.

3. and 4. Group Velocity Problems

Read Feynman Volume I, Section 48-4, "Localized wave trains" carefully. The following two problems (one from Feynman and one from Moore) will give you a little more insight into group velocity.

35.3 The phase velocity of a water wave of wavelength λ is, neglecting surface tension and the effects of finite depth,

$$v_{ph} = \sqrt{g\lambda/2\pi}.$$

(a) Show that the group velocity v_g is one half the phase velocity.

(b) What are the group and phase velocities of a wave of wavelength 1000 m?

Q9D.2 (Requires studying appendix QA.) The eigenfunction corresponding to a definite value p_0 of a quanton's x-momentum is the function

$$\psi_{p_0}(x) = Ae^{ip_0x/\hbar} \quad \text{(Q9.25)}$$

Since a (non-relativistic) free quanton with mass m moving in one dimension with a definite x-momentum p_0 has a definite energy $E_0 = p_0^2/2m$, a free quanton in a definite-momentum state will also be in a definite-energy state. Therefore, by the Time Evolution Rule, its wavefunction as a function of time will be

$$\psi_{p_0}(x, t) = e^{-iEt/\hbar}Ae^{ip_0x/\hbar} = Ae^{i(p_0x-Et)/\hbar} = Ae^{i(kx-\omega t)} \quad \text{(Q9.26)}$$

where $k \equiv p_0/\hbar$ and $\omega \equiv E/\hbar$. If we expand out the exponential, this becomes

$$\psi_{p_0}(x, t) = A[\cos(kx - \omega t) + i\sin(kx - \omega t)] \quad \text{(Q9.27)}$$

which is a sum of traveling waves.

(a) For a traveling wave of this form, the wave's phase speed (the speed at which crests of the wave move) is $|\vec{v}_p| = \omega/k$, as discussed in chapter Q1. Argue that for a nonrelativistic quanton, this phase speed is $p_0/2m$, and that this is *not* equal to the speed a classical particle having this momentum would move.

(b) Argue that empty space is a *dispersive* medium for a free quanton's wavefunction (one in which the phase speed depends on the wave's wavelength).

(c) In a dispersive medium, *information* does not move at the phase speed. For example, if you look at waves created by throwing a rock into a pond, you will see an expanding ring of disturbance through which wave crests move at a speed faster than the ring itself (appearing at the ring's trailing edge and disappearing at the leading edge)! In a dispersive medium, the speed that information moves (that is, the speed of the expanding ring in the case above) is actually the **group speed** $|\vec{v}_g| \equiv d\omega/dk$. Show that the group speed for our wave *is* equal to the speed we would expect for a classical particle having momentum p_0.

(d) For an extremely relativistic quanton such as a photon, the quanton's relativistic energy is related to its momentum by $E_0 = p_0c$. Show that in this case both the phase and group speeds of a photon's quantum wavefunction are equal to the speed of light c.