Special Relativity – Fourth Problem Set

3-11 Addition of Velocities

Before launching into this problem, it is important to make sure we know the <u>definition</u> of velocity, which is thankfully reasonably simple:

$$V_x \equiv \frac{\Delta x}{\Delta t} \qquad \left(\text{contrast with speed } s = |V_x|\right)$$

If there is motion along more than one axis, there is also

$$V_y = \frac{\Delta y}{\Delta t} \quad \text{and} \quad V_z = \frac{\Delta z}{\Delta t} \qquad \left(\text{speed } s = \sqrt{V_x^2 + V_y^2 + V_z^2}\right)$$

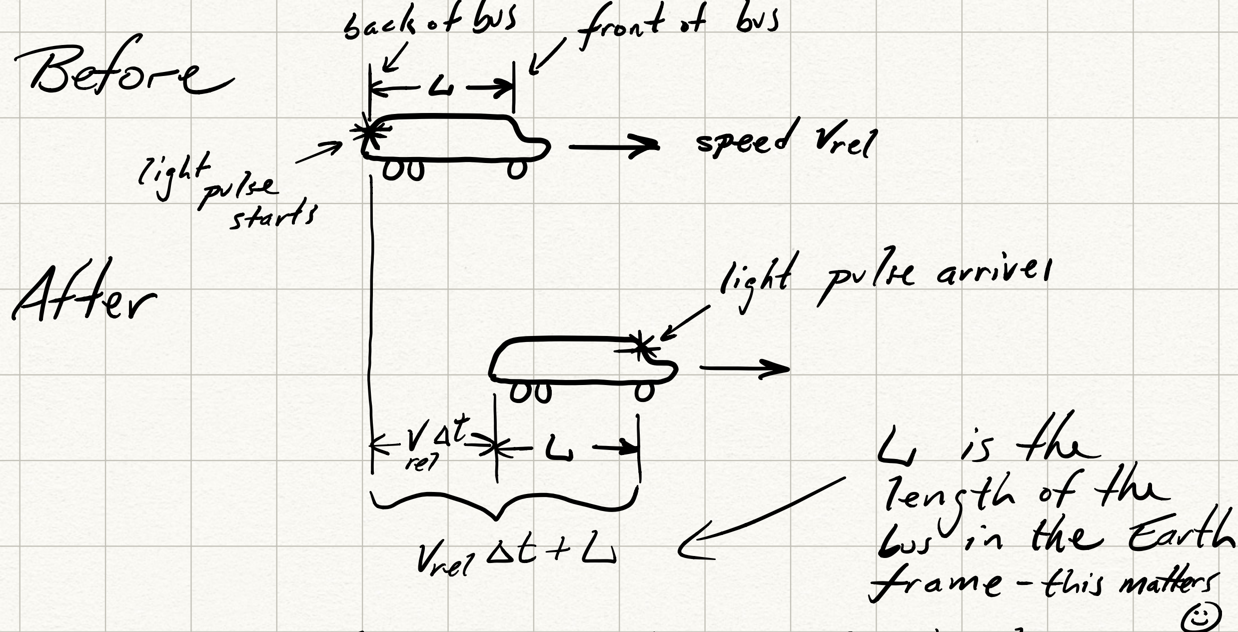
If the particle is moving steadily (with no acceleration), it doesn't matter how long an elapsed time you use, because the distance traveled in any direction is proportional to the time. However, if the particle is accelerating, the velocity is changing, and you have to use as short a time Δt as possible to get an accurate estimate of the velocity.

We capture the idea of "as short a time as possible" by writing

$$v_x \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$ (and similarly for v_y and v_z)

read this as "the limit as Δt goes to zero"

(a) We need a quality diagram



Now the pulse is light, and light always goes the speed of light, so it must be that

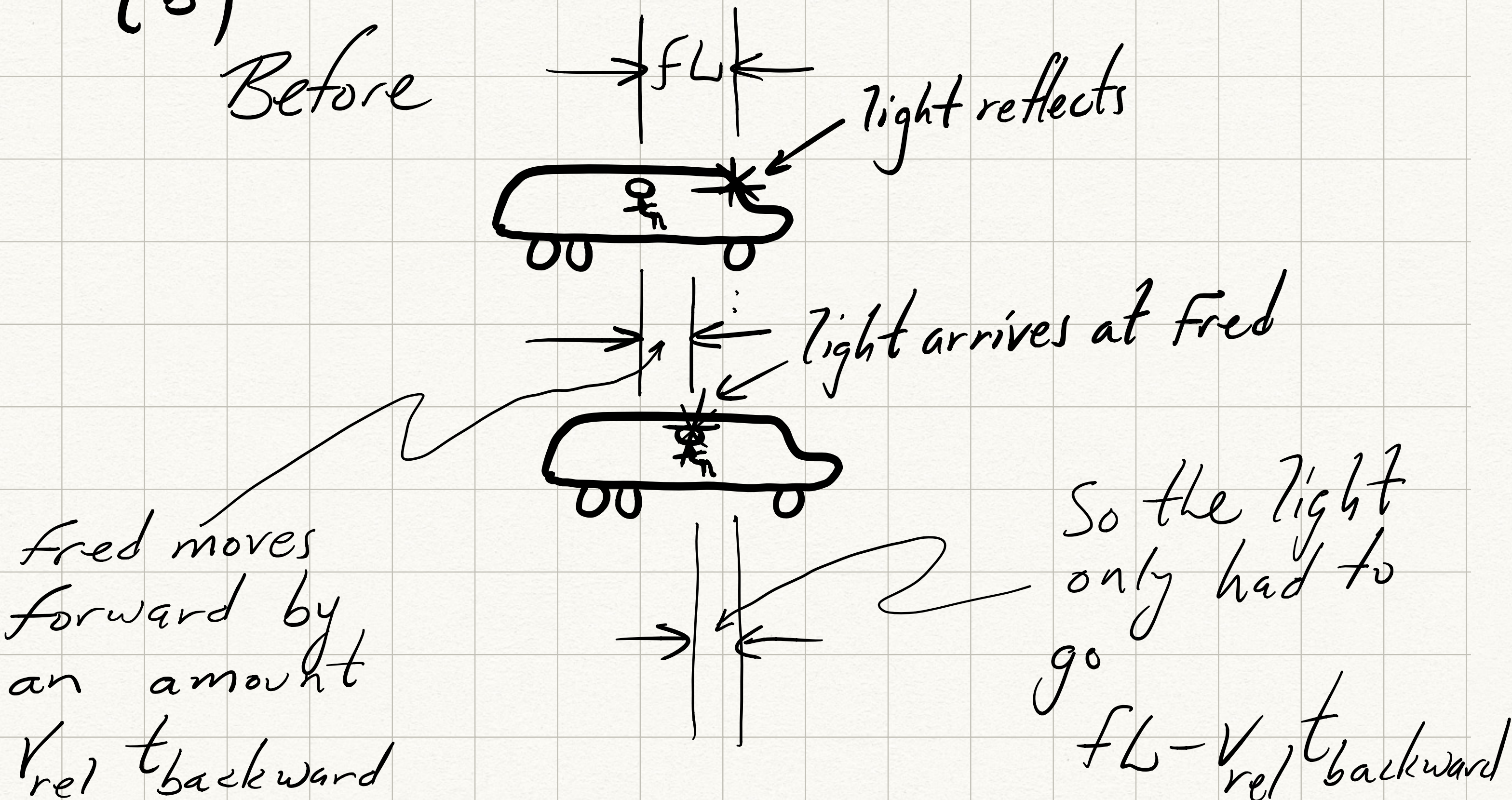
$$v_{rel}\Delta t + L = c\Delta t$$

In our crafty units, $c=1$, and also, the authors called Δt, $t_{forward}$, so we have

$$v_{rel}\, t_{forward} + L = t_{forward} \quad \text{or}$$

$$t_{forward} = \frac{L}{1 - v_{rel}}$$ which is what we were asked to show

(b)



Now light goes $c\, t_{backward}$ in a time $t_{backward}$ (and $c=1$) so

$$fL - v_{rel}\, t_{backward} = c\, t_{backward} = t_{backward}$$

or $$t_{backward} = \frac{fL}{1+v_{rel}}$$

(c) <u>Net</u> distance traveled by light flash is

$c\, t_{forward} - c\, t_{backward} = t_{forward} - t_{backward}$

Meanwhile, the whole time, the bullet has been traveling forward with speed v_{bullet}, so it has traveled $\left(t_{forward} + t_{backward}\right) v_{bullet}$

(c) (CONT'D) By assumption, they meet at Fred, so we must have

$$t_{forward} - t_{backward} = (t_{forward} + t_{backward}) V_{bullet}$$

or

$$t_{forward} (1 - V_{bullet}) = t_{backward} (1 + V_{bullet})$$

(d) Use parts (a) and (b) to get rid of $t_{forward}$ and $t_{backward}$ in the equation in (c):

$$\frac{L}{1 - V_{rel}} (1 - V_{bullet}) = \frac{fL}{1 + V_{rel}} (1 + V_{bullet})$$

Cancel the L's and solve for f

$$f = \frac{1 - V_{bullet}}{1 - V_{rel}} \frac{1 + V_{rel}}{1 + V_{bullet}}$$

(e) Repeat all of (a)–(d) but in the space bus frame. In the spacebus frame $V_{rel} = 0$. However V_{bullet} has some new and different value, so we need a name for it: call it V'_{bullet}. So we get a very similar result for f:

$$f = \frac{1 - V'_{bullet}}{1 + V'_{bullet}}$$

(f) Equate the two expressions for f

$$\frac{1-v_{bullet}}{1-v_{rel}}\,\frac{1+v_{rel}}{1+v_{bullet}} = \frac{1-v'_{bullet}}{1+v'_{bullet}}$$

It seems like we are getting something for nothing! Amazing. Keep going. Solve for v'_{bullet}

$$(1+v'_{bullet})\underbrace{\frac{1-v_{bullet}}{1-v_{rel}}\,\frac{1+v_{rel}}{1+v_{bullet}}}_{goop} = 1-v'_{bullet}$$

or

$$v'_{bullet} = \frac{1-goop}{1+goop} = \frac{1-\frac{1-v_{bullet}}{1-v_{rel}}\,\frac{1+v_{rel}}{1+v_{bullet}}}{1+\frac{1-v_{bullet}}{1-v_{rel}}\,\frac{1+v_{rel}}{1+v_{bullet}}}$$

$$= \frac{(1-v_{rel})(1+v_{bullet})-(1+v_{rel})(1-v_{bullet})}{(1-v_{rel})(1+v_{bullet})+(1+v_{rel})(1-v_{bullet})}$$

$$= \frac{1-v_{rel}+v_{bullet}-v_{rel}v_{bullet}-(1+v_{rel}-v_{bullet}-v_{rel}v_{bullet})}{+}$$

$$= \frac{2(v_{bullet}-v_{rel})}{2-2v_{rel}v_{bullet}} = \frac{v_{bullet}-v_{rel}}{1-v_{bullet}v_{rel}}$$

Those rascals solved for v_{bullet}, but I solved for v'_{bullet}. It is easy to get theirs from mine without more algebra. Just exchange primed with unprimed values

$$v_{bullet} = \frac{v'_{bullet}-v'_{rel}}{1-v'_{bullet}v'_{rel}}$$

But isn't $v'_{rel} = -v_{rel}$

motion of Earth in bus frame

So

$$v_{bullet} = \frac{v'_{bullet}+v_{rel}}{1+v'_{bullet}v_{rel}}$$

(g) (1) $v_{rel} = 108\ \frac{km}{hr} = \frac{108000\ m}{3600\ s} = 30\ \frac{m}{s}$

$$= \frac{30\ \frac{m}{s}}{3\times10^8\ \frac{m}{s}} = 10^{-7} \leftarrow \text{in crafty units}$$

$$v_{bullet} = 600\ \frac{m}{s} = \frac{600\ \frac{m}{s}}{3\times10^8\ \frac{m}{s}} = 2\times10^{-6}$$

$$1 + v_{rel} v_{bullet} = 1 + 10^{-7}\cdot 2\times10^{-6} = 1 + 2\times10^{-13}$$

My calculator can't even tell this as different from 1 (if it has 10-12 sig figs). So pretending

$\frac{1}{1 + v_{rel} v_{bullet}} = 1$, the formula simplifies

to

$$v_{bullet} = v'_{bullet} + v_{rel}$$

That's the Newtonian result. Velocities just add.

(2) $v_{rel} = \frac{3}{4}$, $v'_{bullet} = \frac{3}{4}$

$$\frac{\frac{3}{4} + \frac{3}{4}}{1 + \frac{3}{4}\cdot\frac{3}{4}} = \frac{\frac{6}{4}}{1 + \frac{9}{16}} = \frac{16\cdot\frac{6}{4}}{16 + 9}$$

$$= \frac{24}{25} = 0.96 \quad (\text{not } 1.5!)$$

(3) Bullet going light-speed

$v_{rel} = \frac{3}{4}$, $v'_{bullet} = 1$

$$\frac{1 + \frac{3}{4}}{1 + 1\cdot\frac{3}{4}} = 1 \leftarrow$$ also goes light-speed according to Earth observer

(4) Bullet going backward at light-speed

$V_{rel} = \frac{3}{4}$

$V'_{bullet} = -1$

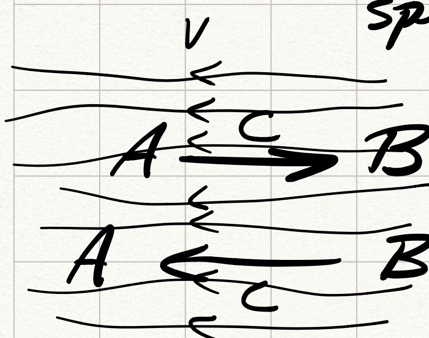
$$V_{bullet} = \frac{V'_{bullet} + V_{rel}}{1 + V'_{bullet} V_{rel}} = \frac{-1 + \frac{3}{4}}{1 + (-1)(\frac{3}{4})} = -1$$

I don't usually look for discussion in problem solutions. However, how did we get something for nothing!?

We analyzed the bullet motion in two frames. We got a result that is caused by both length contraction and time dilation. Our solution (getting V_{bullet} in terms of V'_{bullet}) never had the $\sqrt{1 - V_{rel}^2}$ length contraction nor the $\frac{1}{\sqrt{1 - V_{rel}^2}}$ time dilation as intermediate results.

3-12 The Michelson—Morley Experiment

(a) Round trip time for plane. Part (a) is non-relativistic reasoning. But to get us anticipating relativity, call the airplane's air speed c. The headwind blowing from B to A has speed v. Don't use relativitistic addition of speeds for this part.



← going against the wind, plane's ground speed is $c-v$

← " with " " " " " $c+v$

Assume the distance is d. The plane needs to go a total distance (relative to the ground) of $2d$.

The first leg of the trip takes $\frac{d}{c-v}$.

The second leg takes $\frac{d}{c+v}$.

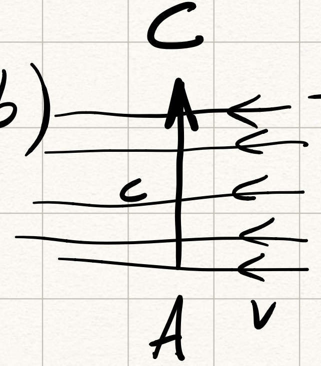
If the air were still, the time would be $\frac{2d}{c}$.

Compute the ratio

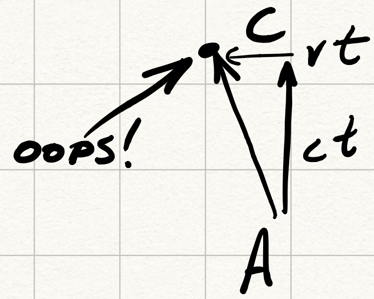
$$\frac{\frac{\cancel{d}}{c-v}+\frac{\cancel{d}}{c+v}}{\frac{2\cancel{d}}{c}} = \frac{c+v+(c-v)}{(c+v)(c-v)\frac{2}{c}}$$

$$= \frac{\cancel{2}c}{(c^2-v^2)\frac{\cancel{2}}{c}} = \frac{1}{(c^2-v^2)\frac{1}{c^2}} = \frac{1}{1-v^2/c^2}$$

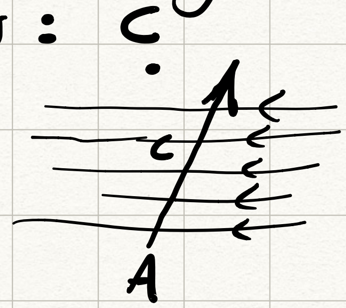
(b)

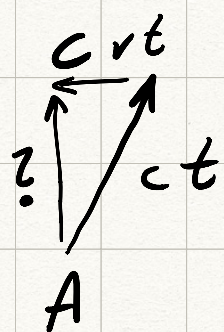


The plane cannot fly straight if it did it would arrive downwind of C



So the diagram must look like this:





What is the side labeled ?. Well yes, it is d, but what else is it? Aha, by the Pythagorean Theorem it is $\sqrt{c^2t^2 - v^2t^2}$.

So $d = \sqrt{c^2t^2 - v^2t^2} = t\sqrt{c^2 - v^2}$

So $t = \dfrac{d}{\sqrt{c^2 - v^2}}$, and the same on the return trip.

Compute the ratio of the round-trip time going perpendicular to the wind to the round-trip time in still air:

$$\frac{\cancel{2}\cdot\frac{\cancel{d}}{\sqrt{c^2-v^2}}}{\cancel{2}\,\frac{\cancel{d}}{c}} = \frac{c}{\sqrt{c^2-v^2}} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

(c) The one going perpendicularly will arrive home first, because $1-\frac{v^2}{c^2}$ is a number less than 1 and $\sqrt{1-v^2/c^2}$ is not as much less than 1. So $\frac{1}{1-v^2/c^2}$ is more than $\frac{1}{\sqrt{1-v^2/c^2}}$.

If $\frac{v}{c} \ll 1$, then $\frac{v^2}{c^2} \ll 1$. We can use the binomial expansion

$$(a+b)^n = a^n + n a^{n-1} b$$

with $a=1$ $b=-v^2/c^2$ and n is $\begin{cases} -1 \text{ for } \frac{1}{1-v^2/c^2} \\ -\frac{1}{2} \text{ for } \frac{1}{\sqrt{1-v^2/c^2}} \end{cases}$

So our approximation for the result in part (a) is $1+(-1)(-v^2/c^2)$

$$= 1 + v^2/c^2$$

And our approximation for the result in part (b) is $1+(-\frac{1}{2})(-v^2/c^2)$

$$= 1 + \tfrac{1}{2} v^2/c^2$$

The difference is $\frac{1}{2} v^2/c^2$

The difference in the round-trip travel time is $\frac{2d}{c}\left(\frac{1}{2}\frac{v^2}{c^2}\right)$

The authors let $L = 2d$, and this is what they call Δt, so

$$\Delta t = \frac{L}{c}\frac{1}{2}\frac{v^2}{c^2}$$ and we agree.

(d) The fussbudget has

$$\Delta t = 4 \text{ sec}$$
$$L = 600 \text{ km}$$
$$c = 300 \frac{\text{km}}{\text{hr}}$$

The discrepancy can be explained by a wind going either with or against the slowest planes (and perpendicular to the fastest planes. The speed of the wind is given by

$$\frac{v^2}{c^2} = \frac{2c\Delta t}{L} = \frac{2 \cdot 300 \frac{\text{km}}{\text{hr}} \, 4\text{se}}{600\text{km}} = \frac{4\text{sec}}{1\text{hr}} = \frac{1}{900}$$

or $\frac{v}{c} = \frac{1}{30}$ or $v = \frac{1}{30} c = \frac{1}{30} 300 \frac{\text{km}}{\text{hr}} = 10 \frac{\text{km}}{\text{hr}}$

(e) Michelson and Morley have

$$L = 22\text{m}$$
$$c = 3 \times 10^8 \frac{\text{m}}{\text{s}}$$
$$v = 30 \times 10^3 \frac{\text{m}}{\text{s}}$$

$\left.\right\}$ v^2/c^2 is $(10^{-4})^2 = 10^{-8}$

So their Δt is

$$\Delta t = \frac{L}{c} \frac{1}{2} \frac{v^2}{c^2} = \frac{22\text{m}}{3 \times 10^8 \text{m/s}} \frac{1}{2} \cdot 10^{-8}$$
$$= \frac{11}{3} \times 10^{-16} \text{ s} \approx 3.7 \times 10^{-16} \text{ s} \approx 4 \times 10^{-4} \text{ ps}$$

Even if time can be measured to picoseconds, that is still more than 1000 times smaller (0.4 femtoseconds).

(f) $f\lambda = c$ and $f \equiv \frac{1}{T}$

$$\Rightarrow \frac{\lambda}{T} = c \Rightarrow T = \frac{\lambda}{c}$$

Plug in $\lambda = 589\,nm \approx 600 \times 10^{-9}\,m$

and $c = 3 \times 10^{8}\,m/s$

$$T \approx \frac{600 \times 10^{-9}\,m}{3 \times 10^{8}\,m/s} = 200 \times 10^{-17}\,s = 2 \times 10^{-15}\,s$$

They apparently could do 100 times this well. So they could detect $\Delta t = 2 \times 10^{-17}\,s$

The Earth's motion results in an expected Δt of $4 \times 10^{-16}\,s$, so they are 40 times as sensitive as needed to detect ether flowing at Earth's orbital speed. Since the effect is proportional to v^2 they actually can detect v $\frac{1}{\sqrt{40}}$ of the orbital speed. They saw nothing.

$\Longrightarrow$ there is no ether drift

$\Longrightarrow$ the speed of light is independent of the motion of the observer.

(g) Discussion: is there a way to evade the conclusion in (f)?