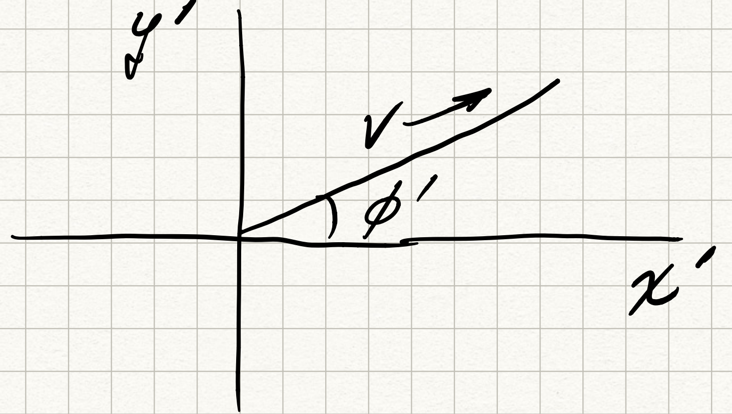
Fifth Problem Set — Your choice of any of 3-13 to 3-17 to present — plus L-9

Well first off I notice that Part (b) of L-9 requires L-8. So maybe we first do that.

L-8 Transformation of velocity direction.

In the rocket frame, a particle moves as follows:



Particle position at t' is

$$x' = vt'\cos\phi'$$
$$y' = vt'\sin\phi'$$

Thanks to the Lorentz transformation, equation L-106, we can mindlessly grind out any t, x, and y if we know t', x', and y'.

$$t = \frac{v_{rel}x' + t'}{\sqrt{1-v_{rel}^2}} = \frac{v_{rel}\, vt'\cos\phi' + t'}{\sqrt{1-v_{rel}^2}}$$

$$x = \frac{x' + v_{rel}t'}{\sqrt{1-v_{rel}^2}} = \frac{vt'\cos\phi' + v_{rel}t'}{\sqrt{1-v_{rel}^2}}$$

$$y = vt'\sin\phi'$$

Now take $\frac{y}{x}$ and define ϕ by $\tan\phi = \frac{y}{x}$

$$\tan\phi = \frac{vt'\sin\phi'}{\frac{vt'\cos\phi' + v_{rel}t'}{\sqrt{1-v_{rel}^2}}}$$ ← cancel the t' in numerator and denominator

$$= \sqrt{1-v_{rel}^2}\,\frac{v\sin\phi'}{v\cos\phi' + v_{rel}}$$

Maybe that is as good an answer as we can get for L-8, and we can go on to L-9.

L-9 The Headlight Effect

We just need to put $v=1$ into what we just found in L-8.

$$\tan\phi = \sqrt{1-v_{rel}^2}\,\frac{\sin\phi'}{\cos\phi' + v_{rel}}$$

Does not look like the desired answer! What is your next step? We want an answer with only cosines. How about squaring both sides, and then using $\tan^2\phi = \frac{\sin^2\phi}{\cos^2\phi} = \frac{1-\cos^2\phi}{\cos^2\phi}$

$$= \frac{1}{\cos^2\phi} - 1$$

Also use $\sin^2\phi' = 1 - \cos^2\phi'$.

Now we only have cosines!

$$\frac{1}{\cos^2\phi} - 1 = (1 - V_{rel}^2)\frac{1-\cos^2\phi'}{(\cos\phi' + V_{rel})^2}$$

$$(1-\cos^2\phi)(\cos\phi' + V_{rel})^2$$
$$= (1-V_{rel}^2)(1-\cos^2\phi')\cos^2\phi$$

$$(\cos\phi' + V_{rel})^2 = (\cos\phi' + V_{rel})^2 \cos^2\phi$$
$$+ (1-V_{rel}^2)(1-\cos^2\phi')\cos^2\phi$$

Let's see if we can simplify all the stuff that multiplies $\cos^2\phi$ on the RHS. It is:

$$\cancel{\cos^2\phi'} + 2V_{rel}\cos\phi' + \cancel{V_{rel}^2} + 1 - \cancel{V_{rel}^2} - \cancel{\cos^2\phi'} + V_{rel}^2\cos^2\phi'$$
$$= 1 + 2V_{rel}\cos\phi' + V_{rel}^2\cos^2\phi'$$
$$= (1 + V_{rel}\cos\phi')^2$$

WOO HOO! A PERFECT SQUARE!

So we have

$$(\cos\phi' + V_{rel})^2 = (1 + V_{rel}\cos\phi')^2\cos^2\phi$$

Take the square root of this equation, and we have

$$\cos\phi = \frac{\cos\phi' + V_{rel}}{1 + V_{rel}\cos\phi'}$$

Exactly what Taylor and Wheeler wanted us to get. ✓

(b) Well, because I used L-8 to get L-9(a) by putting $v=1$, rather than rederiving L-9 from the Lorentz transformation, of course this works. See last page.

(c) They are asking us to put $\phi' = 90°$ into the equation. Call what you get in that special case ϕ_0.

$$\cos\phi_0 = \frac{\cos 90° + V_{rel}}{1 + V_{rel}\cos 90°} = \frac{0 + V_{rel}}{1 + V_{rel}\cdot 0} = V_{rel}$$

✓

My answer to 6 is a little unsatisfying. We could have started L-9 by saying a flashlight flashed at an angle ϕ' satisfies

$$x' = t'\cos\phi'$$

$$y' = t'\sin\phi'$$

Then Lorentz transformed that to get

$$t = \frac{v_{rel}\,(t'\cos\phi') + t'}{\sqrt{1-v_{rel}^2}}$$

$$x = \frac{t'\cos\phi' + v_{rel}\,t'}{\sqrt{1-v_{rel}^2}}$$

$$y = t'\sin\phi'$$

Again define ϕ by

$$\tan\phi = \frac{y}{x} = \frac{t'\sin\phi'}{\frac{t'\cos\phi' + v_{rel}\,t'}{\sqrt{1-v_{rel}^2}}}$$

$$= \sqrt{1-v_{rel}^2}\,\frac{\sin\phi'}{\cos\phi' + v_{rel}}$$

Our equation for $\tan\phi$ is indeed the equation for $\tan\phi$ that we had in L-8 with v set to 1.