Special Relativity — HW 8 Solution

4-1 (a)-(f) with $D = 12$ ly and $v = \frac{12}{13}$

4-1(a) In the Earth frame, the time taken is

$$\frac{D}{v} = \frac{12 \text{ ly}}{12/13} = 13 \text{ ly of time}$$

(or just 13 yrs of time)

(b) We aren't adding any layover, so this is also 13 yrs.

(c) It takes the same amount of time to return, so 13 more yrs, or a total of 26 years to make the round trip.

(d) A few ways of doing this one – mine is <u>not</u> the best <u>and</u> mine was originally wrong!

If $v = \frac{12}{13}$ then $\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{13}{\sqrt{169-144}} = \frac{13}{5}$. So James says the distance to the star is contracted to $\frac{D}{\gamma}$.
If a star is $\frac{D}{\gamma}$ away from you and approaching at speed v, then it will get to you in a time

$$\frac{D}{\gamma v} = \frac{12 \text{ ly}}{\frac{\cancel{13}}{5} \cdot \frac{12}{\cancel{13}}} = \frac{12 \text{ ly}}{\frac{12}{5}} = 5 \text{ ly} = 5 \text{ years}$$

(e) As in (b), we arent adding any layover.

(f) It takes the same amount of time to return, so 5 more years, or a total of 10 years to make the round trip. 26 yrs − 10 yrs = 16 yrs, and so James has aged 16 yrs less than those who remained on Earth.

Inspired by 4-2: The Quintuplets

Event 1: Q_E and Q_R are born on Earth

frame	time	position
E	0	0
R	0	0
V	0	$-D$
ME	0	$-2D$
MR	$-2D\gamma' v$	$-2D\gamma'$

EASY WAY The mechanical way of filling in these slots:

Apply L-11a to the ME-frame coordinates of the event, using $v_{rel} = -v$

$$t' = +v\gamma'(-2D) + \gamma' \cdot 0$$
$$= -2D\gamma' v$$
$$x' = \gamma'(-2D) + v \cdot \gamma' \cdot 0$$
$$= -2D\gamma'$$

Why can you get away with using L-11a? Because the ME frame and the MR frame have coincident origins.

HARD WAY

The more sophisticated way of filling in the two slots — use invariance of the interval.

Since ME and MR have the same origin, the interval^2 from their origins to Event 1 must be the same. From the ME frame we have

$$\text{interval}^2 = 0^2 - (-2D)^2$$
$$= -4D^2$$

We must also have

$$-4D^2 = t'^2 - x'^2 \quad *$$

where t' and x' are the MR coordinates of Event 1.

That is not enough info. We have yet to use that MR is moving in the $-x$ direction relative to ME at speed v.

One way to use that extra bit of information is to use that the distance from E to ME is Lorentz-contracted to $2D/\gamma'$ according to MR. Then, if t' is an amount T in the past, at which time MR is vT to the right of ME, we have

$$\left.\begin{aligned} t' &= -T \\ x' &= -2D/\gamma' - vT \end{aligned}\right\}$$

Stick that set of facts into Equation (*)

Solve quadratic, get $T = 2D\gamma' v$

$t' = -2D\gamma' v \quad x' = -2D\gamma'$ ✓✓

WHEW!

Event 2: Q_{ME} and Q_{MR} are born at Mirror Earth

frame	time	position
E	0	2D
R	$2D\gamma' v$	$2D\gamma'$
V	0	D
ME	0	0
MR	0	0

It's pretty obvious from Event 1 that the blanks are going to get filled in as

$2D\gamma' v$ and $2D\gamma'$

However, let's do it by noticing that the origins of E and R are coincident, and so we can use L-11a with $v_{rel} = v$ to get t' and x'.

$$t' = v\gamma' 2D + \gamma' \cdot 0 = 2D\gamma' v \quad \checkmark\checkmark$$

$$x' = \gamma' 2D - v\gamma' \cdot 0 = 2D\gamma' \quad \checkmark\checkmark$$

Event 3: Q_R and Q_{MR}, going opposite directions, pass Q_V at Vega

frame	time	position
E	D/v	D
R	$D/(\gamma' v)$	0
V	D/v	0
ME	D/v	$-D$
MR	$D/(\gamma' v)$	0

There are lots of ways to get the two missing times. They are the same. Invariance of the interval says

$$t'^2 - 0^2 = \frac{D^2}{v^2} - D^2 = D^2\left(\frac{1}{v^2} - 1\right) = \frac{D^2}{v^2}(1 - v^2) = \frac{D^2}{v^2\gamma'^2}$$

So the missing times are $D/(\gamma' v)$.

Q_V is $\frac{D}{v}$ and Q_R and M_R are $D/(\gamma' v)$. The difference is $D(1 - 1/\gamma')/v$.

Event 4: Q_{MR} arrives at Earth and meets Q_E

frame	time	position
E	$2D/V$	0
R	$2\gamma D/V$	$-2D\gamma$
V	$2D/V$	$-D$
ME	$2D/V$	$-2D$
MR	$2D/(\gamma V)$	0

This row I used L-11a with $V_{rel}=V$

$$t' = V\gamma \cdot 0 + \gamma \cdot \frac{2D}{V} = \gamma \frac{2D}{V}$$

$$x' = \gamma \cdot 0 - \cancel{V}\gamma \frac{2D}{\cancel{V}} = -2D\gamma$$

This blank is easy to get with invariance of the interval.

$$\left(\frac{2D}{V}\right)^2 - (-2D)^2 = t'^2 - 0^2 \implies t' = 2D\sqrt{\frac{1}{V^2}-1}$$

$$t' = \frac{2D}{V}\sqrt{1-V^2} = 2D/(\gamma V)$$

Q_E is $\frac{2D}{V}$ and Q_{MR} is $\frac{2D}{\gamma V}$

So Q_E is older than Q_{MR} by

$$\frac{2D}{V} - \frac{2D}{\gamma V} = \frac{2D}{V}\left(1-\frac{1}{\gamma}\right)$$

Event 5: Q_R arrives at Mirror Earth and meets Q_{ME}

frame	time	position
E	$2D/V$	$2D$
R	$2D/(\gamma V)$	0
V	$2D/V$	D
ME	$2D/V$	0
MR	$2\gamma D/V$	$2\gamma D$

This row we get with L-11a and $V_{rel}=-V$

$$t' = V\gamma \cdot 0 + \gamma\, 2D/V = 2\gamma D/V$$

$$x' = \gamma \cdot 0 + \cancel{V}\gamma \cdot 2D/\cancel{V} = 2\gamma D$$

This blank we get with invariance of the interval and get $2D/(\gamma V)$

Q_{ME} is $2D/V$ and Q_R is $2D/(\gamma V)$ so

$$Q_{ME} - Q_R = \frac{2D}{V}\left(1-\frac{1}{\gamma}\right)$$

Repeat with $D = 25\,ly$ $\quad v = \frac{4}{5}$ $\quad \gamma = \frac{1}{\sqrt{1-0.64}} = \frac{1}{0.6} = \frac{5}{3}$

Event 1

frame	time	position
E	0	0
R	0	0
V	0	$-25\,ly$
ME	0	$-50\,ly$
MR	-67 yrs	$-83\,ly$

$$-2(25\,ly)\cdot\frac{5}{3} = \frac{-250}{3}\,ly = -83\,ly$$

$$-2(25\,ly)\frac{\cancel{5}}{3}\cdot\frac{4}{\cancel{5}} = -50\,ly\cdot\frac{4}{3} = -67\,ly = -67 \text{ years}$$

Event 2

frame	time	position
E	0	$50\,ly$
R	67 yrs	$83\,ly$
V	0	$25\,ly$
ME	0	0
MR	0	0

Event 3

frame	time	position
E	31 yrs	$25\,ly$
R	19 yrs	0
V	31 yrs	0
ME	31 yrs	$-25\,ly$
MR	19 yrs	0

Age difference is $12\frac{1}{2}$ yrs

Event 4

frame	time	position
E	62 yrs	0
R	104 yrs	$-83\,ly$
V	62 yrs	$-25\,ly$
ME	62 yrs	$-50\,ly$
MR	37 yrs	0

Age difference is 25 yrs

Event 5

frame	time	position
E	62 yrs	$50\,ly$
R	37 yrs	0
V	62 yrs	$25\,ly$
ME	62 yrs	0
MR	104 yrs	$83\,ly$

Age difference is 25 yrs