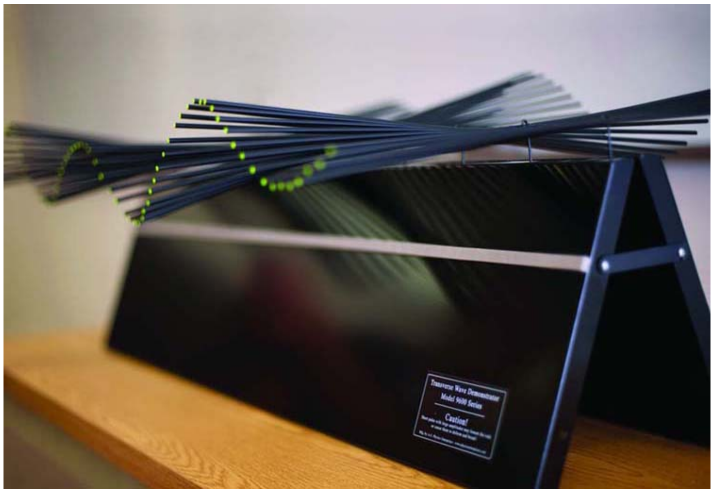
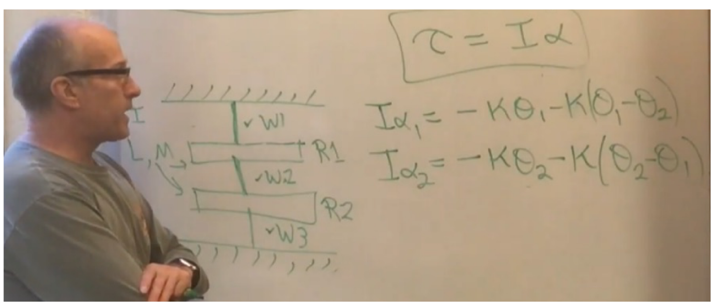
Torsion Waves



In Monday's class we saw waves on the above device. By the end of today's class, I hope you will have some idea of how those waves arise from the equations you already know.

The above device has 75 rods connected by 74 wires and is almost 1 meter long. 75 is too much to think about initially, so in the flipped mini-lecture, I started with one rod and one wire.

Then once I solved that, I went on to two rods and three wires:



We call these coupled differential equations, because as you can see the derivative of θ_1 involves θ_2 and the derivative of the θ_2 involves θ_1.

$$I\,\frac{d^2\,\theta_1}{dt^2} + \kappa\theta_1 + \kappa(\theta_1 - \theta_2) = 0$$

$$I\,\frac{d^2\,\theta_2}{dt^2} + \kappa\theta_2 + \kappa(\theta_2 - \theta_1) = 0$$

Now as usual, one way to solve a differential equation is to guess the solution and see if it works, and in the mini-lecture, I explained why two guesses might work.

Rods Twisting Together

One guess is that the rods twist together:

$\theta_1(t) = A \cos \omega t$
$\theta_2(t) = A \cos \omega t$

In the space below, put the above guess into the differential equations. Does it have a chance of working? If it is going to work, what must ω be?

Rods Twisting Oppositely

Another guess is that the rods twist oppositely:

$\theta_1(t) = B \cos \omega' t$
$\theta_2(t) = -B \cos \omega' t$

Put that guess into the differential equations. Does it have a chance of working? If it is going to work, what must ω' be?

75 Rods

Now that you have some experience with two coupled differential equations. Let's see if we can set up 75 coupled differential equations!

The first rod is special because it is only connected on one side:

$$I\frac{d^2\theta_1}{dt^2} + \kappa(\theta_1 - \theta_2) = 0$$

The 75th rod is also special, because it is only connected on one side:

$$I\frac{d^2\theta_{75}}{dt^2} + \kappa(\theta_{75} - \theta_{74}) = 0$$

All the rods in between are like each other, because they each have a connection to the rod on either side:

$$I\frac{d^2\theta_i}{dt^2} + \kappa(\theta_i - \theta_{i-1}) + \kappa(\theta_i - \theta_{i+1}) = 0$$

Just so that I'm sure you know what an equation like that means, plug in $i = 33$, and write out what that equation says about the 33rd rod:

You see how we are now stuck with 75 coupled differential equations instead of just two? If not, write out the equation for the 34th rod.

A Very Expensive Classroom Demonstration

That demonstration is a $400 device. But let's assume Saint Mary's has $400,000 to spend on a better demonstration and they lined up thousands of these, so that instead of having 75 rods and 92cm, you have 75,000 rods stretching almost a kilometer.

Check this out — Virginia Tech (phys.vt.edu) connected two of them together:

In fact maybe Saint Mary's has an infinite budget for this demonstration and our demonstration goes on forever!

Then there is nothing special about any of the rods. Every rod is connected to one on either side. And that makes the problem easier! (I told you the ends were hard. They cause the reflections and that we just aren't going to get into that.)

To summarize, in our infinite demonstration, every rod obeys:

$$I \frac{d^2 \theta_i}{dt^2} + \kappa(\theta_i - \theta_{i-1}) + \kappa(\theta_i - \theta_{i+1}) = 0$$

and *i* doesn't just go from 1 to 75, it now goes from -∞ to ∞.

You saw that the system has waves moving to the right and waves moving to the left. Let's guess that there are solutions of the form:

$$\theta_i(t) = A \cos(k x_i - \omega t)$$

x_i is x-position of the *i*th rod. If the rods have spacing *s* and we put the 0th rod at x=0, then

$$x_i = i\, s$$

$k = \frac{2\pi}{\lambda}$ is one property of our guess, and $\omega = 2\pi f$ is another, and there is the overall multiplier *A*.

We don't really need λ and f, I'm just reminding you of them. Note that our problem has both *k*s and *κ*s ("kays" and "kappas"), and they have nothing much to do with each other, so you'll need to be neat and keep the two distinguished.

To summarize, here is our guess:

$$\theta_i(t) = A \cos(k i s - \omega t)$$

Your turn. Plug the guess into the equation that every rod obeys. Use scratch paper. Divide the problem up into two parts. Part I is to calculate:

$$I \frac{d^2 \theta_i}{dt^2}$$

Part II is to calculate:

$$\kappa(\theta_i - \theta_{i-1}) + \kappa(\theta_i - \theta_{i+1})$$

Simplifying that is going to involve some nasty-looking but not really that hard trig identities.

Once you've the above two parts all simplified down on scratch paper, then add the first and second part together and set it equal to zero. Does the guess work? (Yes, or we wouldn't have done all this!) What is the relationship between k and ω that must be true for the guess to work?

If k were very small (to be more precise, if $k s \ll 1$) that corresponds to a long wavelength. If that is a good approximation, use the Taylor series expansion for cosine to simplify your answer. Using that $v = \omega / k$ what have you found for v for long wavelengths?

Now that you are done with this derivation, you have a decent idea of where Knight Equation 16.1 comes from. However, after all this work you have just done, Knight Section 16.4 is still going to be pretty difficult, because Knight uses more math than I did above. But at least you have some idea of how each segment of a chain or a string can affect the next segment, and therefore how waves on a string arise from Newton's laws.