MECHANICS

THIRD EDITION

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Forced Harmonic Oscillator
Excerpt from Symon, Mechanics, 3rd Edition

Assignment for Monday, 10/2/2019 is:

1. Townsend 4.6
2. Townsend 3.2
3. Townsend 4.8
4. Symon #50

ADDISON-WESLEY PUBLISHING COMPANY
Reading, Massachusetts
Menlo Park, California · London · Amsterdam · Don Mills, Ontario · Sydney

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The behavior is shown in Fig. 2.5 for the case of a system displaced from equilibrium and released ($x_0 \neq 0$, $v_0 = 0$). The reader should draw similar curves for the case where the system is given a sharp blow at $t = 0$ (i.e., $x_0 = 0$, $v_0 \neq 0$).

2.10 THE FORCED HARMONIC OSCILLATOR

The harmonic oscillator subject to an external applied force is governed by Eq. (2.91). In order to simplify the problem of solving this equation, we state the following theorem:

Theorem 3. *If $x_i(t)$ is a solution of an inhomogeneous linear equation* [*e.g.*, Eq. (2.91)], *and $x_h(t)$ is a solution of the corresponding homogeneous equation* [*e.g.*, *Eq.* (2.90)], *then $x(t) = x_i(t) + x_h(t)$ is also a solution of the inhomogeneous equation.*

This theorem applies whether the coefficients in the equation are constants or functions of t. The proof is a matter of straightforward substitution, and is left to the reader. In consequence of Theorem 3, if we know the general solution x_h of the homogeneous equation (2.90) (we found this in Section 2.9), then we need find only one particular solution x_i of the inhomogeneous equation (2.91). For we can add x_i to x_h and obtain a solution of Eq. (2.91) which contains two arbitrary constants and is therefore the general solution.

The most important case is that of a sinusoidally oscillating applied force. If the applied force oscillates with angular frequency ω and amplitude F_0, the equation of motion is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos(\omega t + \theta_0), \tag{2.148}$$

where θ_0 is a constant specifying the phase of the applied force. There are, of course, many solutions of Eq. (2.148), of which we need find only one. From physical considerations, we expect that one solution will be a steady oscillation of the coordinate x at the same frequency as the applied force:

$$x = A_s \cos(\omega t + \theta_s). \tag{2.149}$$

The amplitude A_s and phase θ_s of the oscillations in x will have to be determined by substituting Eq. (2.149) in Eq. (2.148). This procedure is straightforward and leads to the correct answer. The algebra is simpler, however, if we write the force as the real part of a complex function:*

$$F(t) = \mathrm{Re}(\mathbf{F}_0 e^{i\omega t}), \tag{2.150}$$

$$\mathbf{F}_0 = F_0 e^{i\theta_0}. \tag{2.151}$$

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*Note the use of bold face type ($\mathbf{F}$, $\mathbf{x}$) to distinguish complex quantities from the corresponding real quantities (F, x).

Thus if we can find a solution $\mathbf{x}(t)$ of

$$m\frac{d^2\mathbf{x}}{dt^2}+b\frac{d\mathbf{x}}{dt}+k\mathbf{x} = \mathbf{F}_0 e^{i\omega t}, \tag{2.152}$$

then, by splitting the equation into real and imaginary parts, we can show* that the real part of $\mathbf{x}(t)$ will satisfy Eq. (2.148). We assume a solution of the form

$$\mathbf{x} = \mathbf{x}_0 e^{i\omega t},$$

so that

$$\dot{\mathbf{x}} = i\omega\mathbf{x}_0 e^{i\omega t}, \qquad \ddot{\mathbf{x}} = -\omega^2\mathbf{x}_0 e^{i\omega t}. \tag{2.153}$$

Substituting in Eq. (2.152), we solve for $\mathbf{x}_0$:

$$\mathbf{x}_0 = \frac{\mathbf{F}_0/m}{\omega_0^2-\omega^2+2i\gamma\omega}. \tag{2.154}$$

The solution of Eq. (2.152) is therefore

$$\mathbf{x} = \mathbf{x}_0 e^{i\omega t} = \frac{(\mathbf{F}_0/m)e^{i\omega t}}{\omega_0^2-\omega^2+2i\gamma\omega}. \tag{2.155}$$

The simplest way to write Eq. (2.155) is to express the denominator in polar form [Eq. (2.109)]:

$$\omega_0^2-\omega^2+2i\gamma\omega = [(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}\exp\left(i\tan^{-1}\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right). \tag{2.156}$$

It is convenient to define the angle

$$\beta = \frac{\pi}{2}-\tan^{-1}\frac{2\gamma\omega}{\omega_0^2-\omega^2} = \tan^{-1}\frac{\omega_0^2-\omega^2}{2\gamma\omega}, \tag{2.157}$$

$$\sin\beta = \frac{\omega_0^2-\omega^2}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}}, \tag{2.158}$$

$$\cos\beta = \frac{2\gamma\omega}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}}. \tag{2.159}$$

This definition is purely a matter of taste, and is arranged so that $\beta = 0$ when $\omega = \omega_0$ and $\beta \to \pm\pi/2$ as $\omega \to \pm\infty$. (See Fig. 2.6.) This definition also makes our treatment parallel to the customary treatment of Eq. (2.92) in electrical engineering. If we use Eqs. (2.156) and (2.157) and the fact that

$$i = e^{i\pi/2}, \tag{2.160}$$

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*The assertion "we can show that . . . " throughout this book will mean that the reader who has followed the discussion to this point should be able to supply the proof himself. (In this case, put $\mathbf{x} = x+iy$ and the result falls out.) Long or tricky proofs will either be given in the text, or a reference cited, or the reader will be warned that it is not easy.

we may rewrite Eq. (2.155) in the form

$$\mathbf{x} = \frac{F_0}{im[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}}\, e^{i(\omega t+\theta_0+\beta)}. \tag{2.161}$$

The complex velocity is

$$\dot{\mathbf{x}} = i\omega\mathbf{x} = \frac{\omega F_0}{m[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}}\, e^{i(\omega t+\theta_0+\beta)}. \tag{2.162}$$

The real position and velocity are then

$$x = \mathrm{Re}(\mathbf{x})$$
$$= \frac{F_0}{m}\,\frac{1}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}}\sin(\omega t+\theta_0+\beta), \tag{2.163}$$

and

$$\dot{x} = \mathrm{Re}(\dot{\mathbf{x}})$$
$$= \frac{F_0}{m}\,\frac{\omega}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}}\cos(\omega t+\theta_0+\beta). \tag{2.164}$$

This is a particular solution of Eq. (2.148) containing no arbitrary constants. By Theorem 3 and Eq. (2.133), the general solution (for the underdamped oscillator) is

$$x = Ae^{-\gamma t}\cos(\omega_1 t+\theta)+\frac{F_0/m}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}}\sin(\omega t+\theta_0+\beta). \tag{2.165}$$

This solution contains two arbitrary constants A, θ, whose values are determined by the initial values x_0, v_0 at $t = 0$. The first term dies out exponentially in time and is called the *transient*. The second term is called the *steady state*, and oscillates with constant amplitude. The transient depends on the initial conditions. The steady state which remains after the transient dies away is independent of the initial conditions. (When there is no damping, $\gamma = 0$, the "transient" does not die away, but we may still define it as that part of the solution which has the natural frequency $\omega_1 = \omega_0$; the term "transient" is not very descriptive in this case.)

In the steady state, the rate at which work is done on the oscillator by the applied force is

$$\dot{x}F(t) = \frac{F_0^2}{m}\,\frac{\omega}{[(\omega^2-\omega_0^2)^2+4\gamma^2\omega^2]^{1/2}}\cos(\omega t+\theta_0)\cos(\omega t+\theta_0+\beta)$$

$$= \frac{F_0^2}{m}\,\frac{\omega\cos\beta\cos^2(\omega t+\theta_0)}{[(\omega^2-\omega_0^2)^2+4\gamma^2\omega^2]^{1/2}}-\frac{F_0^2}{2m}\,\frac{\omega\sin\beta\sin 2(\omega t+\theta_0)}{[(\omega^2-\omega_0^2)^2+4\gamma^2\omega^2]^{1/2}}, \tag{2.166}$$

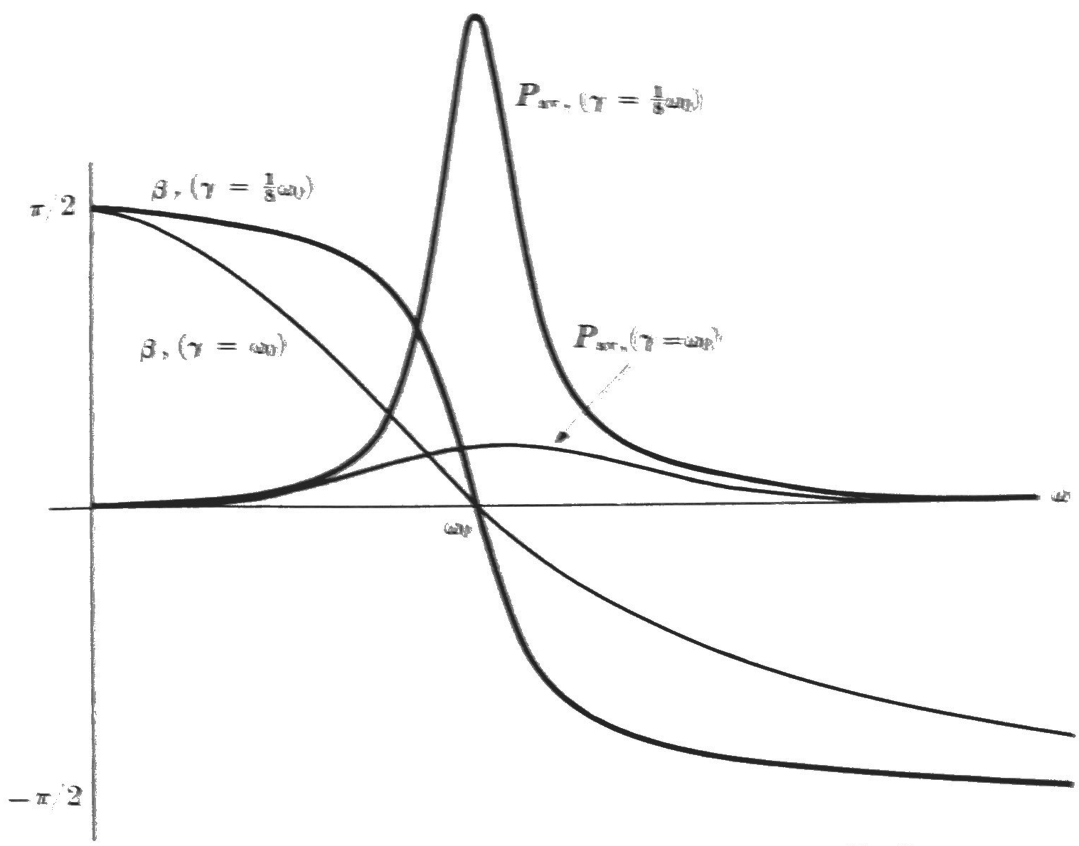

Fig. 2.6 Power and phase of forced harmonic oscillations.

The last term on the right is zero on the average, while the average value of $\cos^2(\omega t+\theta_0)$ over a complete cycle is $\frac{1}{2}$. Hence the average power delivered by the applied force is

$$P_{av} = \langle \dot{x}F(t)\rangle_{av} = \frac{F_0^2 \cos\beta}{2m} \frac{\omega}{[(\omega^2-\omega_0^2)^2+4\gamma^2\omega^2]^{1/2}}. \tag{2.167}$$

or

$$P_{av} = \tfrac{1}{2}F_0\dot{x}_m \cos\beta. \tag{2.168}$$

where $\dot{x}_m$ is the maximum value of $\dot{x}$. A similar relation holds for power delivered to an electrical circuit. The factor $\cos\beta$ is called the *power factor*. In the electrical case, β is the phase angle between the current and the applied emf. Using formula (2.162) for $\cos\beta$, we can rewrite Eq. (2.167):

$$P_{av} = \frac{F_0^2}{m} \frac{\gamma\omega^2}{(\omega^2-\omega_0^2)^2+4\gamma^2\omega^2}. \tag{2.169}$$

It is easy to show that in the steady state power is supplied to the oscillator at the same average rate that power is being dissipated by friction, as of course it must be. The power P_{av} has a maximum for $\omega = \omega_0$. In Fig. 2.6, the power P_{av} (in arbitrary units) and the phase β of steady-state forced oscillations are plotted against ω for two values of γ. The heavy curves are for small damping; the light curves are for greater damping. Formula (2.169) can be simplified somewhat in

Find a particular solution by expressing F as the real part of a complex exponential function and looking for a solution for x having the same exponential time dependence.

47. An undamped harmonic oscillator ($b = 0$), initially at rest, is subject beginning at $t = 0$ to an applied force $F_0 \sin \omega t$. Find the motion $x(t)$.

48. An undamped harmonic oscillator ($b = 0$) is subject to an applied force $F_0 \cos \omega t$. Show that if $\omega = \omega_0$, there is no steady-state solution. Find a particular solution by starting with a solution for $\omega = \omega_0 + \varepsilon$, and passing to the limit $\varepsilon \to 0$. [*Hint:* If you start with the steady-state solution and let $\varepsilon \to 0$, it will blow up. Try starting with a solution which fits the initial condition $x_0 = 0$, so that it cannot blow up at $t = 0$.]

49. A critically damped harmonic oscillator with mass m and spring constant k, is subject to an applied force $F_0 \cos \omega t$. If, at $t = 0$, $x = x_0$ and $v = v_0$, what is $x(t)$?

50. A force $F_0 \cos(\omega t + \theta_0)$ acts on a damped harmonic oscillator beginning at $t = 0$.
a) What must be the initial values of x and v in order that there be no transient?
b) If instead $x_0 = v_0 = 0$, find the amplitude A and phase θ of the transient in terms of F_0, θ_0.

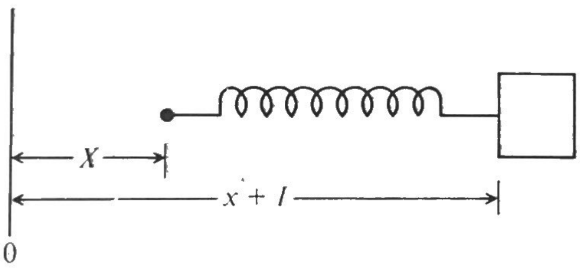

Fig. 2.11

51. A mass m is attached to a spring with force constant k, relaxed length l, as shown in Fig. 2.11. The left end of the spring is not fixed, but is instead made to oscillate with amplitude a, frequency ω, so that $X = a \sin \omega t$, where X is measured from a fixed reference point 0. Write the equation of motion, and show that it is equivalent to Eq. (2.148) with an applied force $ka \sin \omega t$, if the friction is given by Eq. (2.31). Show that, if the friction comes instead from a dashpot connected between the ends of the spring, so that the frictional force is $-b(\dot{x} - \dot{X})$, then the equation of motion has an additional applied force $\omega ba \cos \omega t$.

52. An automobile weighing one ton (2000 lb, including passengers but excluding wheels and everything else below the springs) settles one inch closer to the road for every 200 lb of passengers. It is driven at 20 mph over a washboard road with sinusoidal undulations having a distance between bumps of 1 ft and an amplitude of 2 in (height of bumps and depth of holes from mean road level). Find the amplitude of oscillation of the automobile, assuming it moves vertically as a simple harmonic oscillator without damping (no shock absorbers). (Neglect the mass of wheels and springs.) If shock absorbers are added to provide damping, is the ride better or worse? (Use the result of Problem 51.)