Homework 15 due Monday, 11/4 is to complete this handout (see all the places marked "Your turn"), and also do Townsend 5.1 and 5.6.

From last time:

The four states in the direct product basis are:

$|\uparrow\rangle_1 \otimes |\uparrow\rangle_2$

$|\uparrow\rangle_1 \otimes |\downarrow\rangle_2$

$|\downarrow\rangle_1 \otimes |\uparrow\rangle_2$

$|\downarrow\rangle_2 \otimes |\downarrow\rangle_2$

We are just going to number these

$|1\rangle$

$|2\rangle$

$|3\rangle$

$|4\rangle$

The simplest Hamiltonian one can come up with for the interaction of two spins is

$$H = \frac{2A}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2 \quad \leftarrow \text{Townsend 5.9}$$

The 2 and the $\frac{1}{\hbar^2}$ are just in there for convenience and so that A has dimensions of energy.

In the direct product basis $\vec{S}_1 \cdot \vec{S}_2$ can be written

$$S_{1x} \otimes Id_2 \cdot Id_1 \otimes S_{2x}$$
$$+ S_{1y} \otimes Id_2 \cdot Id_1 \otimes S_{2y}$$
$$+ S_{1z} \otimes Id_2 \cdot Id_1 \otimes S_{2z}$$

but that simplifies to

$$S_{1x} \otimes S_{2x} + S_{1y} \otimes S_{2y} + S_{1z} \otimes S_{2z}$$

and you have to know that that is what $\vec{S}_1 \cdot \vec{S}_2$ is short for.

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that's as far as we got on 10/30

Now we want to find the matrix of the Hamiltonian in the $|1\rangle, |2\rangle, |3\rangle, |4\rangle$ basis.

Recall that S_{1x} and S_{1y} are kind of a mess in the z-basis (and of course S_{2x} and S_{2y}). One way of dealing with this is to introduce the raising and lowering operators

$$S_{1+} = S_{1x} + iS_{1y} \qquad S_{1-} = S_{1x} - iS_{1y}$$

$$S_{2+} = S_{2x} + iS_{2y} \qquad S_{2-} = S_{2x} - iS_{2y}$$

YOUR TURN

Find an expression for $H = \frac{2A}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2$ that only uses $S_{1+}, S_{1-}, S_{1z}, S_{2+}, S_{2-}$, and S_{2z}

YOUR TURN AGAIN

Find

$\langle 1|H|1\rangle$

YOUR TURN AGAIN

$\langle 1|H|2\rangle$, $\langle 1|H|3\rangle$, $\langle 1|H|4\rangle$, and $\langle 2|H|1\rangle$ are all 0.

Find $\langle 2|H|2\rangle$

YOUR TURN AGAIN

Find $\langle 2|H|3\rangle$

If things have gone well, you now know 7 of the entries of $\langle i|H|j\rangle$ and the matrix looks like

$$\begin{pmatrix} A/2 & 0 & 0 & 0 \\ 0 & -A/2 & A & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix}$$

Really we still have nine more entries to go, but by now you know how to get all of them and I'll just give you the whole 4x4 matrix for H

$$H \xrightarrow[\substack{\text{in the} \\ |1\rangle,|2\rangle,|3\rangle,|4\rangle \\ \text{basis}}]{} \begin{pmatrix} A/2 & 0 & 0 & 0 \\ 0 & -A/2 & A & 0 \\ 0 & A & -A/2 & 0 \\ 0 & 0 & 0 & A/2 \end{pmatrix}$$

This is Townsend 5.14.

<u>YOUR TURN AGAIN</u> What are the eigenvectors and eigenvalues of

$$H\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = E\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Use the usual method. This will involve a determinant of a 4x4 matrix. It will be a 4th order polynomial with four roots and four eigenvectors. Normalize the eigenvector