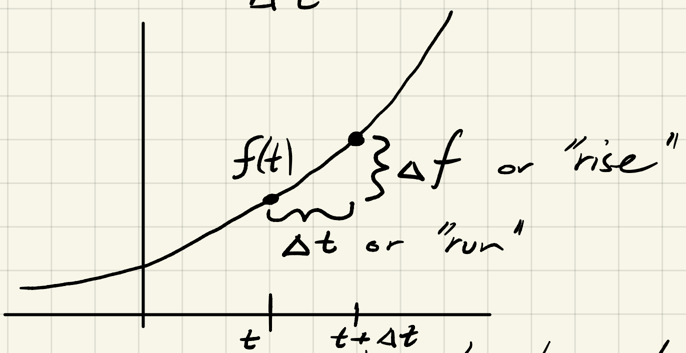
RC Circuits

Read my primer on exponentials to recall the most important properties of exponentials. The one that plays a huge role in RC circuits is that if $f(t)$ is an exponential (such as 2^t, 10^t, or e^t, or even $e^{\alpha t}$) then

$\frac{\Delta f}{\Delta t}$ is proportional to $f(t)$



This is only approximately true! It becomes closer and closer to being exactly true as Δt gets smaller and smaller.

The mathematical notation for $\frac{\Delta f}{\Delta t}$ as Δt gets infinitesimally small is $\frac{df}{dt}$, and $\frac{df}{dt}$ has the interpretation of being the slope of the tangent line to $f(t)$ at the point $(t, f(t))$.

So the exact version of "$\frac{\Delta f}{\Delta t}$ is proportional to $f(t)$" is actually $\frac{df}{dt}$ is proportional to $f(t)$.

For 2^t the proportionality constant is $\ln 2$

$$\frac{d\, 2^t}{dt} = \ln 2 \cdot 2^t$$

For 10^t the proportionality constant is $\ln 10$

$$\frac{d\, 10^t}{dt} = \ln 10 \cdot 10^t$$

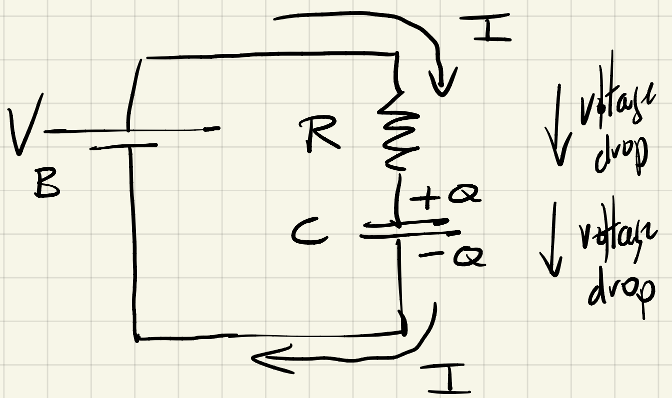
For e^t the proportionality constant is 1.
That's what makes e so special.

$$\frac{d\,e^t}{dt} = e^t$$

It is a function whose slope is always itself!

For $e^{\alpha t}$
$$\frac{d\,e^{\alpha t}}{dt} = \alpha e^{\alpha t}$$

Consider the following circuit



The charge is piling up on one of the capacitor's plates, and the opposite charge is forming on the other.

We refine $I \equiv \frac{Q}{t}$

to something more accurate: $I \equiv \frac{\Delta Q}{\Delta t}$

Then, because the current I might be varying we make a further refinement:

$$I(t) \equiv \frac{dQ}{dt}$$

The voltage across the resitor, by Ohm's Law is
$$V_R(t) = I(t)\,R$$

The voltage across the capacitor is $V_c(t) = \frac{Q(t)}{C}$

the equation involving all the voltages is
$$V_B - I(t)R - \frac{Q(t)}{C} = 0$$

Let's imagine that the circuit has been sitting for a long time. The capacitor can't keep charging forever so the current must stop. That means the voltage across the resistor must go to 0 (because $V_R(t) = I(t)R$), and so the eqn.

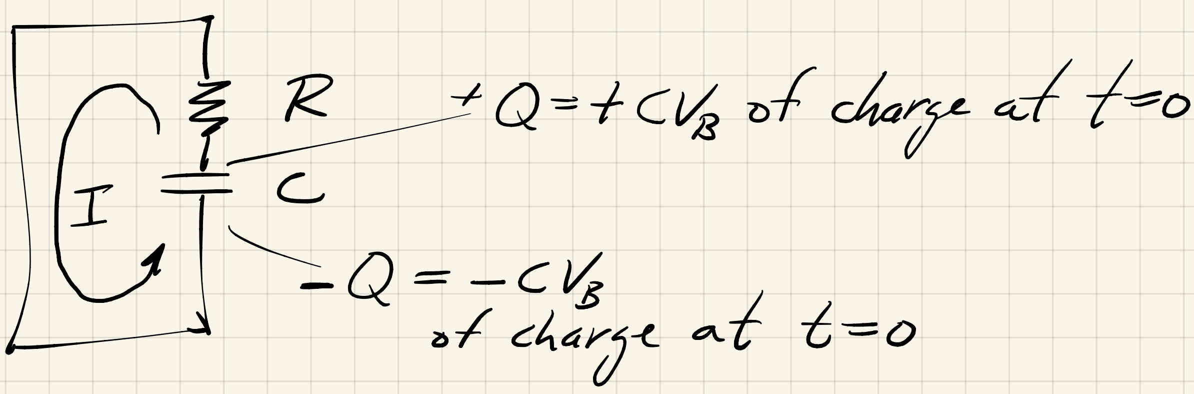
$$V_B = I(t)R(t) + \frac{Q(t)}{C}$$

after a very long time must simplify to

$$V_B = \frac{Q(t)}{C}$$

In other words, eventually $Q(t)$ settles down to CV_B.

Let's imagine a different situation: the charge on the capacitor has settled down to CV_B and then at $t=0$, the battery is removed and the circuit simplified to



Now the current I is going to go the other way. With I oriented counterclockwise and Q labeled as shown $\frac{dQ}{dt} = -I(t)$

The equation during discharge of the capacitor is

$$\frac{Q(t)}{C} - I(t)R = 0$$

Substitute in $\quad I(t) = -\frac{dQ}{dt}$

$$\frac{Q(t)}{C} + \frac{dQ}{dt}R = 0$$

Or

$$\frac{dQ}{dt} = \frac{-1}{RC}Q(t)$$

This is the equation an exponential obeys, with $\quad \alpha = -\frac{1}{RC}$

So

$$Q(t) = Q(0)e^{-t/RC}$$

But $Q(0) = CV_B$, so we can put that in and get

$$Q(t) = Q(0)e^{-t/RC}$$

The charge on a capacitor discharging through a resistor is a decaying exponential.

The original situation is also a "decaying" exponential, but it decays towards CV_B as $t \to \infty$. The solution is

$$Q(t) = CV_B\left(1 - e^{-t/RC}\right)$$

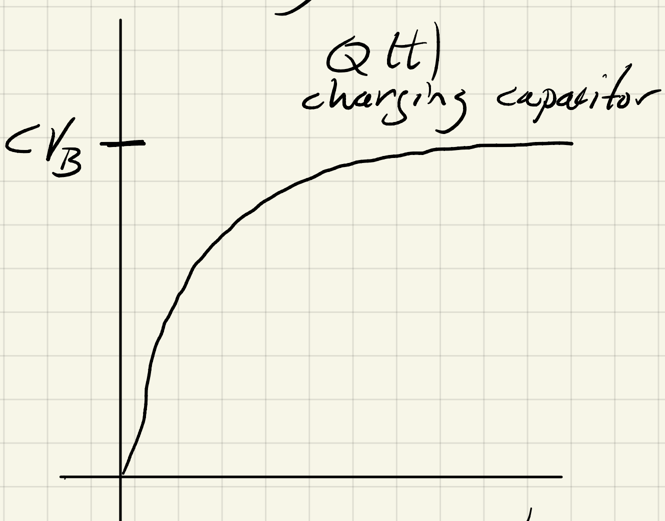
In class, the excellent question was asked, 'how long does this decay toward the final value take in the real world? Let's use some R and C values we are familiar with: 220Ω and 100μF.

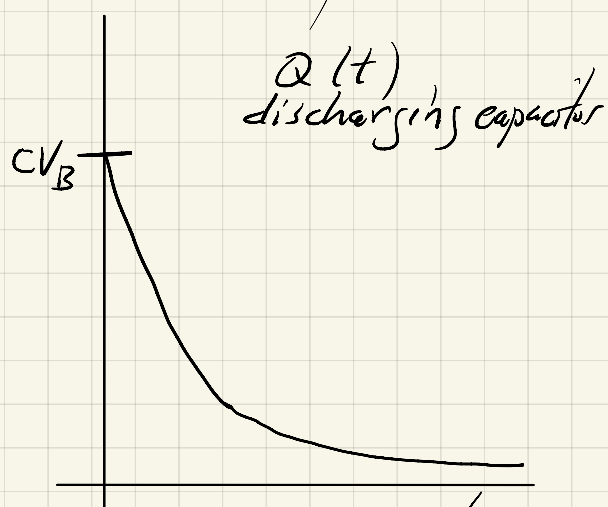
$$RC = 220\,\Omega \cdot 100 \times 10^{-6}\,F$$

$$= 220\,\frac{\cancel{V}}{A} \cdot 100 \times 10^{-6}\,\frac{C}{\cancel{V}}$$

$$= 2.2 \times 10^{-2}\,\frac{C}{A} = 0.022\,s$$

That's about $\frac{1}{50}$ of a second. So after a few $\frac{1}{50}$ s (say $\frac{3}{50}$ s or $\frac{4}{50}$ s) almost all the charging or discharging (depending on the situation) has occurred.





with all this, perhaps you are in a good position to see how the "decoupling capacitor" is charged up and able to quickly provide some current to the servo motor in your last circuit.